MATEMATIKA ÉS FIZIKA

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CONNECTIONS OF THE DIFFERENT MODELS OF THE HYPERBOLIC PLANE

Abstract: The article deals with the isomorphism between some of the different models of the hyperbolic plane and gives an organised system of the models by the help of projections. It provides a more simple discussion of a polar-coordinate model, one of the newest models of the hyperbolic plane.

Introduction

The hyperbolic (Bolyai-Lobachevski) geometry was one of the greatest mathematical inventions in the 19th century that had great effect on the development of geometry, mathematics and physics. Different models were constructed for the introduction of the hyperbolic geometry and for the proof of relative consistency of the axioms. One of the newest models is the polar-coordinate model ([*WIEGAND,* 1992]).

The first chapter contains a short description of the models of the hyperbolic plane. We define the points, lines, incidence, order and reflection in a line in the models. In the second chapter we give a more simple discussion of the polar-coordinate model by the help of an orthogonal projection. In the third chapter we show the isomorphism of the models in an organised system using different projections. And finally, in the fourth chapter we show metric connections between some of the models.

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I. Models of the hyperbolic plane

I.1. *The Cayley–Klein model* ($\mathcal{C}^2$). Let an open unit disk be given in the Euclidean plane where we denote the circle by k^2. The points of the model are the inner points, the lines are the open chords of the disk. Incidence and order correspond to the Euclidean incidence and order. Reflection in line $\boldsymbol{t}$ corresponds to a central collineation whose axis is $\boldsymbol{t}$ and centre is pole of $\boldsymbol{t}$ with respect to the base circle (k^2) and assigns itself to k^2 ([*SZÁSZ*,1973]).

I.2. *The Klein–Poincaré model* ($\mathcal{K}^2$). Let an open unit disk be given in the Euclidean plane where we denote the circle by k^2. The points of the model are the inner points of the disk, the lines are those parts of the lines or circles perpendicular to k^2 which belong to the disk. Incidence and order correspond to the Euclidean incidence and order. If $\boldsymbol{t}$ is a segment (in Euclidean sense) then reflection in the line $\boldsymbol{t}$ is the Euclidean reflection in the line $\boldsymbol{t}$, if $\boldsymbol{t}$ is an arc of circle then reflection in line $\boldsymbol{t}$ is an inversion ([*SZÁSZ*,1973]).

I.3. *The Poincaré's hemisphere model* ($\mathcal{G}^2$). Let an open unit hemisphere with the base circle a^2 in the Euclidean space the given. We denote it by G^2. The points of the model are the points of G^2, the lines are those parts of the circles perpendicular to a^2 that belong to G^2. Incidence and order correspond to the Euclidean incidence and order. Reflection in the line $\boldsymbol{t}$ is a central collineation whose fixed-plane is [$\boldsymbol{t}$] (it denotes the Enclidean plane that contains **t** as a Enclidean curve), centre is pole of [$\boldsymbol{t}$] with respect to G^2 where G^2 is an invariant surface (but not fixed) ([*SZÁSZ*,1973]).

I.4. *The Weierstrass model* ($\mathcal{W}^2$). Let us consider the "upper sheet" of a 2-sheeted hyperboloid given by the equation $x^2+y^2-z^2=-k^2$ and $z>0$ (k is a non-negative constant of the hyperbolic plane). We denote it by H^2, its asymptotic cone by A^2. Let the points of the hyperbolic plane be the points of the surface H^2. Let lines be the intersections of H^2 with the Euclidean planes passing through the point $\mathbf{O}(0,0,0)$ ([*FABER*,1983]). Each intersection is a branch of a hyperbola (*Fig. 1*). A point $\mathbf{P}$ lies on a line $\boldsymbol{l}$ if $\mathbf{P}$ lies on a branch of the hyperbola $\boldsymbol{l}$ in Euclidean sense. Incidence and order in this model are the same as in Euclidean geometry. The ideal points (in Euclidean sense) of the surface will be called the ideal points of the hyperbolic plane. Therefore, each line has two ideal points.

Two lines are called parallel if they have the same ideal point. It means that the intersection of the Euclidean planes determined by the hyperbolic lines is a generator of A^2.

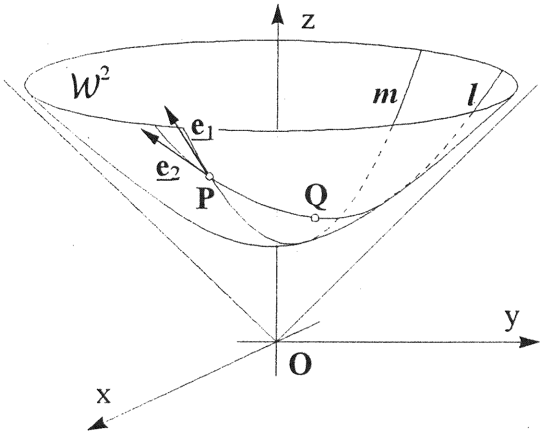

Figure 1

Let the points $\mathbf{P}(p_x,p_y,p_z)$ and $\mathbf{Q}(q_x,q_y,q_z)$ be given. We define the h-inner product of these points (position vectors) by the equation $<\mathbf{P},\mathbf{Q}>= p_x\cdot q_x+p_y\cdot q_y-p_z\cdot q_z$. These points (position vectors) are h-orthogonal if $<\mathbf{P},\mathbf{Q}>=0$.

The equation of a given line $\boldsymbol{m}$ is $<\mathbf{X},\underline{m}>=0$, where $\mathbf{X}\in W^2$ is a running point, the vector $\underline{m}$ is h-normal to the line $\boldsymbol{m}$ (the vector $\underline{m}$ is h-orthogonal to $[\boldsymbol{m}]$ and $<\underline{m},\underline{m}>=k^2$).

The distance between the points $\mathbf{P}$ and $\mathbf{Q}$ is that non-negative d_{PQ} value which satisfies the equation $\cosh \frac{d_{PQ}}{k} = -\frac{1}{k^2}<\mathbf{P},\mathbf{Q}>$.

We define the angle φ of the lines $\boldsymbol{l}$ and $\boldsymbol{m}$ by the equation $\cos\varphi = \frac{<\underline{e}_1,\underline{e}_2>}{\sqrt{<\underline{e}_1,\underline{e}_1>\cdot<\underline{e}_2,\underline{e}_2>}}$, where the vectors $\underline{e}_1$ and $\underline{e}_2$ are the tangent vectors at the point of intersection of lines $\boldsymbol{l}$ and $\boldsymbol{m}$ (*Fig. 1*) and $<\underline{e}_1,\underline{e}_1>=k^2$, $<\underline{e}_2,\underline{e}_2>=k^2$. The expression $\cos^2\varphi = \frac{<\underline{l},\underline{m}>^2}{<\underline{l},\underline{l}>\cdot<\underline{m},\underline{m}>}$ also determines the angle φ of these two lines, where $\underline{l}$ and $\underline{m}$ are h-orthogonal to the lines.

From this point on let the constant k of the Weierstrass model be equal to 1.

Definition. Let the line $\boldsymbol{t}\in W^2$ be given. The lines of intersection of $[\boldsymbol{t}]$ and A^2 are the generators $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ of the cone. Let the line of intersection of the tangent planes (E_1, E_2) determined by these generators of the cone be $\boldsymbol{m}$. The reflection in the line $\boldsymbol{t}$ is defined as a planar affinity with direction $\boldsymbol{m}$, fixed-plane $[\boldsymbol{t}]$ and the ratio of the affinity is equal to -1 (*Fig. 2*).

It can be simply proved that the above planar affinity assigns itself to H^2 and the definition satisfies the properties of the reflection in a line.

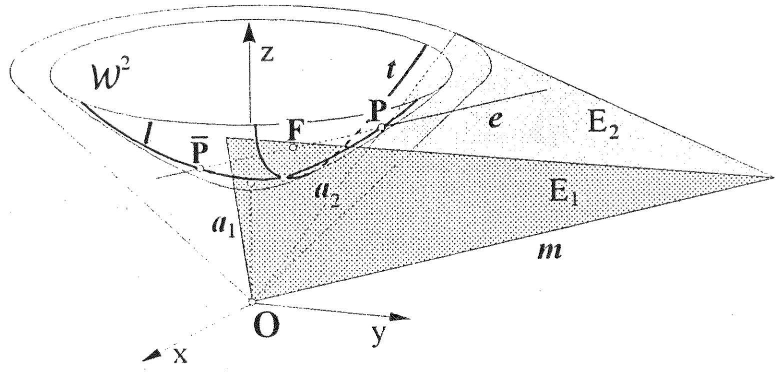

Figure 2

I.5. *The polar-coordinate model* ($\mathcal{P}^2$). We shall denote in the following by $\mathbb{H}^2$ the hyperbolic plane and by $\mathbb{E}^2$ the Euclidean plane. We consider an $(\mathbf{O}_h, x_h, y_h)$ rectangular coordinate system in $\mathbb{H}^2$ and the $(\mathbf{O}_e, x_e, y_e)$ rectangular coordinate system in $\mathbb{E}^2$. Let us assign point $\mathbf{P}_e \in \mathbb{E}^2$ whose polar coordinates are $(\sinh p, \varphi)$ to the point $\mathbf{P}_h \neq \mathbf{O}_h \in \mathbb{H}^2$ with polar coordinates (p, φ). The point $\mathbf{O}_e$ corresponds to the point $\mathbf{O}_h$. This assignment is a bijection between the planes $\mathbb{H}^2$ and $\mathbb{E}^2$ because of the invertability of the function hyperbolic sine. Hence we can model the hyperbolic geometry in the plane $\mathbb{E}^2$ ([*WIEGAND*, 1992]).

The elements of $\mathcal{P}^2$ will be defined in chapter II.

II. A simple discussion of our polar-coordinate model

We give a simple discussion of our polar-coordinate model by the help of the Weierstrass model.

***Lemma* II.1**. Let the orthogonal projections of **K**, $\mathbf{P}\in\mathcal{W}^2$ be **O**, $\mathbf{P}'\in\mathcal{P}^2$. If the distance between **P** and **K** in the Weierstrass model is equal to d, then the Euclidean distance between **P'** and **O** is equal to *sinhd*.

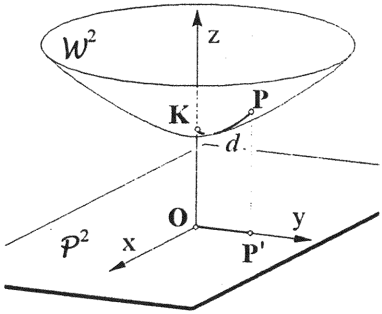

Figure 3

Proof. As the surface H^2 has rotational symmetry (in Euclidean sense) about the z-axis, we may rotate it so, that the yz-plane would contain **P** and **P'**. Thus the orthogonal projection of $\mathbf{P}(0,p_y,\sqrt{p_y^2+1})$, $p_y>0$ is $\mathbf{P}'(0,p_y,0)$ (*Fig. 3*).

The Euclidean distance between the points **O** and **P** is p_y, and the distance d between the points **K** and **P** is given by the expression

$$\cosh d = -\langle \mathbf{K},\mathbf{P}\rangle = \sqrt{p_y^2+1}.$$

Using $\cosh^2 x - \sinh^2 x = 1$ we obtain, that:

$$\sinh d = \sinh(\operatorname{arcosh}\sqrt{p_y^2+1}) = \sqrt{\cosh^2(\operatorname{arcosh}\sqrt{p_y^2+1})-1} = \sqrt{p_y^2+1-1} = p_y \;\square$$

Theorem II.1. The orthogonal projection of the model $\mathcal{W}^2$ onto the xy-plane is the model $\mathcal{P}^2$.

Proof. Let us take the coordinate system in $\mathcal{W}^2$ whose orthogonal projection onto the xy-plane is the (**O**,x,y) rectangular coordinate system. Then the centres (**K**(0,0,1) and **O**) of the models and the coordinate systems correspond to each other.

The angle of any two lines passing through **K** in $\mathcal{W}^2$ is equal to the Euclidean angle of the orthogonal projections of the lines.

Thus using the lemma II.1. the point **P'** with polar coordinates $(\sinh p,\varphi)$ is assigned to a point $\mathbf{P}\in\mathcal{W}^2$ with polar coordinates (p,φ) under the orthogonal projection. $\square$

Corollary. As the orthogonal projection onto the xy-plane is a bijection between the points of the surface H^2 and the xy-plane, the projection with the direction z of the model $\mathcal{P}^2$ defined in the xy-plane with the centre **O** onto the surface H^2 is the model $\mathcal{W}^2$.

Lemma II.2. If we rotate a hyperbola with a semitransverse axis a and a semiconjugate axis b about its transverse axis, we get a two-sheeted hyperboloid. Each intersection of this hyperboloid with the planes passing through its centre is a hyperbola with a semiconjugate axis b.

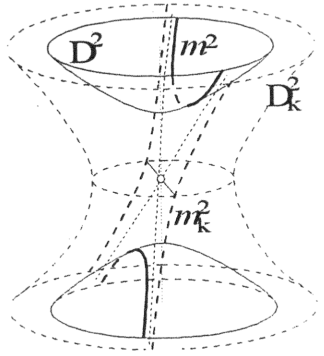

Figure 4

Proof. Let d^2 be a hyperbola with semitransverse axis a and semiconjugate axis b and let its conjugate hyperbola be d_k^2. If we rotate d^2 and d_k^2 about the transverse axis of d^2, we get a two-sheeted hyperboloid D^2 and a one-sheeted hyperboloid D_k^2 which have a common asymptotic cone (*Fig. 4*). The intersection of D^2 and a plane passing through the centre of D^2 is a hyperbola m^2. The intersection of the same plane and D_k^2 is a hyperbola m_k^2, whose transverse axis is the diameter of the throat circle of D_k^2. The hyperbolas m^2 and m_k^2 are conjugate hyperbolas. Since the semitransverse axis of m_k^2 is b, the semiconjugate axis of m^2 is b too. □

Theorem II.2. Lines in the model $\mathcal{P}^2$ are Euclidean lines passing through **O** and branches of hyperbolas whose asymptotes pass through **O** and semiconjugate axis is equal to 1. Each branch of hyperbola and line through **O** is a line in the model $\mathcal{P}^2$.

Proof. Lines of $\mathcal{W}^2$ are such branches of hyperbolas of H^2 whose planes and asymptotes pass through **O**.

If the Enclidean plane of line $l \in \mathcal{W}^2$ is perpendicular to the xy-plane, then its orthogonal projection l' is a line that passes through **O** in the xy-plane.

If the Enclidean plane of l is not perpendicular to the xy-plane then its orthogonal projection l' will be a branch of a hyperbola. Since the orthogonal projection preserves parallelism and incidence, the projections a_1', a_2' of the half asymptotes a_1, a_2 of line l ($z \geq 0$) will be the half asymptotes of l' and contain the point **O** (*Fig. 5*).

Further, the length of the semiconjugate axis of l is equal to 1 because of the lemma II.2. This semiconjugate axis is in the xy-plane since its orthogonal projection is itself. Because of the preservation of incidence it is the semiconjugate axis of l' as well and equal to 1. Any branch of the hyperbola l' with centre **O** and semiconjugate axis 1 in the xy-plane can be considered as the orthogonal projection of a line $l \subset \mathcal{W}^2$. □

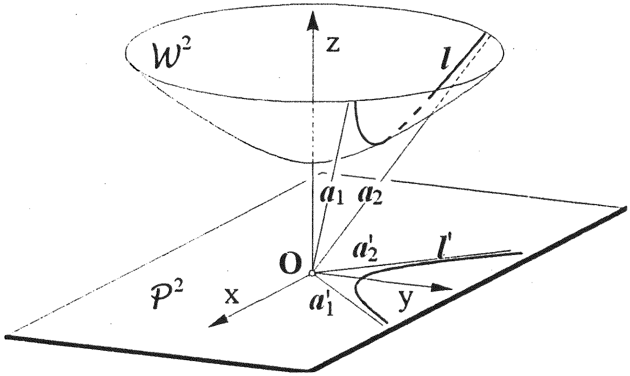

Figure 5

Remark. The equation of a line l' can be determined analytically from the equation of line l.

II.2 Incidence, order and parallelism

Since the orthogonal projection preserves incidence and order, and the orthogonal projection between $\mathcal{P}^2$ and $\mathcal{W}^2$ preserves points and lines, we can define incidence, order and parallelism in $\mathcal{P}^2$ as the projection of $\mathcal{W}^2$.

Definition. Point $\mathbf{P}'\in\mathcal{P}^2$ lies on a line $l'\subset\mathcal{P}^2$ if $\mathbf{P}'$ lies on the curve l' in Euclidean sense. The $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ points are collinear in $\mathcal{P}^2$, $\mathbf{C}'$ is between $\mathbf{A}'$ and $\mathbf{B}'$ if $\mathbf{C}'$ is between $\mathbf{A}'$ and $\mathbf{B}'$ on a branch of a hyperbola in Euclidean sense.

Corollary. Incidence and order in the models $\mathcal{W}^2$ and $\mathcal{P}^2$ can be assigned to each other bijectively.

Definition. The lines, as branches of hyperbolas, are called parallel in $\mathcal{P}^2$ if they have a common half asymptote, or one of the lines is an asymptote of the other.

II.3 Reflection in a line

Definition. Let the line $t' \subset \mathcal{P}^2$ and the point $\mathbf{P}' \in \mathcal{P}^2$ be the orthogonal projections of the line $t \subset \mathcal{W}^2$ and the point $\mathbf{P} \in \mathcal{W}^2$. The image of $\mathbf{P}$ under the reflection in line t is the $\overline{\mathbf{P}} \in \mathcal{W}^2$. Then the image of $\mathbf{P}'$ under the reflection in line t' is the projection of $\overline{\mathbf{P}}$.

It can be proved that the above definition satisfies the properties of the reflection in a line by using the properties of the orthogonal projection.

Remark. If $t' \subset \mathcal{P}^2$ passes through $\mathbf{O}$ then the Euclidean reflection in the Euclidean line t' corresponds to the reflection in the line t'.

Corollary. The reflection in a line in $\mathcal{W}^2$ and $\mathcal{P}^2$ can be assigned to each other bijectively by the help of an orthogonal projection onto the xy-plane.

Definition. Two segments, angels, configurations are called congruent if one can be transformed into the other one by finite number of reflections in a line.

II.4 Circles, horocycles and equidistant curves

The circles, horocycles and equidistant curves in the model $\mathcal{W}^2$ are the plane sections of the surface H^2 ([*FABER*, 1983]). The orthogonal projections of these conic sections give the corresponding curves of $\mathcal{P}^2$.

The circles of $\mathcal{W}^2$ are those plane sections of the surface that are circles or ellipses. The orthogonal projections of these circles onto the xy-plane are such circles, or ellipses whose line of major axis passes through the point $\mathbf{O}$ (in case of circles the point $\mathbf{O}$ is the centre of the circles).

The horocycles of $\mathcal{W}^2$ are those plane sections of the surface that are parabolas. The orthogonal projections of these parabolas are parabolas, whose axes pass through the point $\mathbf{O}$.

The equidistant curves of a line $l \subset \mathcal{W}^2$ are those plane sections of the surface H^2 that are branches of hyperbolas whose planes do not contain $\mathbf{O}$ and are parallel to the plane of the line. The orthogonal projections of these branches of hyperbolas are branches of hyperbolas whose asymptotes do not contain $\mathbf{O}$, but their half asymptotes are parallel to the half asymptotes of l'

and their transverse axis passes through **O**; or those lines that do not contain **O**.

We note that not all the ellipses, parabolas and branches of hyperbolas in the plane $\mathcal{P}^2$ which satisfy the above conditions will be circles, horocycles and equidistant curves. The equations of these configurations can be easily determined by the help of the orthogonal projection.

II.5 Distance

We define the non-negative distance d_{PQ} between the two arbitrary points $\mathbf{P}(p_x,p_y,p_z)$ and $\mathbf{Q}(q_x,q_y,q_z)\in\mathcal{W}^2$ by the equation $\cosh d_{PQ} = -\langle\mathbf{P},\mathbf{Q}\rangle$.

We get $p_x^2+p_y^2-p_z^2=-1$ from $\langle\mathbf{P},\mathbf{P}\rangle = -1$. It follows $p_z=\sqrt{p_x^2+p_y^2+1}$ because of $p_z>0$. We obtain $q_z=\sqrt{q_x^2+q_y^2+1}$ from $\langle\mathbf{Q},\mathbf{Q}\rangle = -1$.

Then $\cosh d_{PQ}= -\langle\mathbf{P},\mathbf{Q}\rangle = -p_xq_x-p_yq_y+p_zq_z =$
$-p_xq_x-p_yq_y+\sqrt{(p_x^2+p_y^2+1)(q_x^2+q_y^2+1)}$.

The orthogonal projections of the points **P** and **Q** are $\mathbf{P}'(p_x,p_y,0)$ and $\mathbf{Q}'(q_x,q_y,0)$, respectively. Only the first and the second coordinates of the points occur in the above equation.

Definition. The non-negative distance $d_{P'Q'}$ between the points $\mathbf{P}'(p_x,p_y,0)\in\mathcal{P}^2$ and $\mathbf{Q}'(q_x,q_y,0)\in\mathcal{P}^2$ is

$$\cosh d_{P'Q'} = -p_xq_x-p_yq_y+\sqrt{(p_x^2+p_y^2+1)(q_x^2+q_y^2+1)}.$$

The above definition of distance really satisfies the properties of distance.

II.6 Angle

Let the vectors $\underline{u}$ and $\underline{v}$ be the tangent vectors at the point of intersection **P** of the lines $l\subset\mathcal{W}^2$ and $m\subset\mathcal{W}^2$, respectively, where $\langle\underline{u},\underline{u}\rangle = \langle\underline{v},\underline{v}\rangle = 1$. We will choose the tangent vectors of the surface so, that their third coordinates would be non-negative. The final points of the vectors $\underline{u}$ and $\underline{v}$ from the point **P** are the points of the asymptotes of ***l*** and ***m*** respectively.

The angle φ between the lines $\boldsymbol{l}$ and $\boldsymbol{m}$ was defined by the following expression: $cos\varphi_{lm} = \langle \underline{u},\underline{v} \rangle$. Since the orthogonal projection preserves incidence, the vectors $\underline{u}'$ and $\underline{v}'$ (the orthogonal projections of $\underline{u}$ and $\underline{v}$) will be the tangent vectors at the point of intersection of the orthogonal projections $\boldsymbol{l}'$ and $\boldsymbol{m}'$ (*Fig. 6*). If the coordinates of $\underline{u}$ and $\underline{v}$ are (u_x,u_y,u_z) and (v_x,v_y,v_z), then the coordinates of $\underline{u}'$ and $\underline{v}'$ are $(u_x,u_y,0)$ and $(v_x,v_y,0)$, respectively. Since $\langle \underline{u},\underline{u} \rangle = u_x^2+u_y^2-u_z^2=1$ and $u_z \geq 0$, therefore $u_z=\sqrt{u_x^2+u_y^2-1}$. Similarly, we get $v_z=\sqrt{v_x^2+v_y^2-1}$.

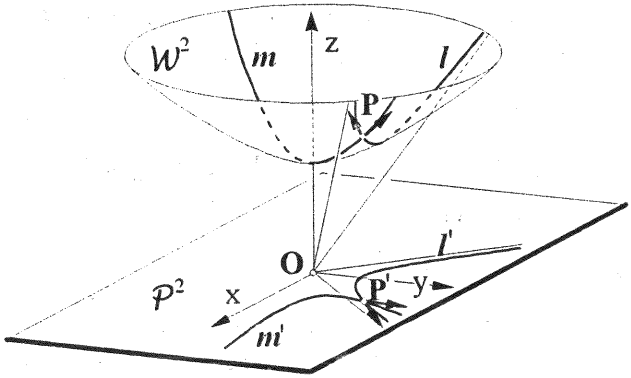

Figure 6

Then it holds $cos\varphi_{lm} = \langle \underline{u},\underline{v} \rangle = u_xv_x + u_yv_y - \sqrt{\left(u_x^2+u_y^2-1\right)\left(v_x^2+v_y^2-1\right)}$ for the angle of the lines $\boldsymbol{l}$ and $\boldsymbol{m}$.

Definition. The angle $\varphi_{l'm'}$ of lines $\boldsymbol{l}' \subset \mathcal{P}^2$ and $\boldsymbol{m}' \subset \mathcal{P}^2$ is

$$cos\varphi_{l'm'} = u_xv_x + u_yv_y - \sqrt{\left(u_x^2+u_y^2-1\right)\left(v_x^2+v_y^2-1\right)},$$

where the vectors $\underline{u}'$ $(u_x,u_y,0)$ and $\underline{v}'$ $(v_x,v_y,0)$ are the tangent vectors of the lines $\boldsymbol{l}'$ and $\boldsymbol{m}'$ at their point of intersection **P'** and they are the orthogonal projections of the tangent vectors (that satisfy the above conditions) of the corresponding lines in $\mathcal{W}^2$. These vectors are located from **P'** to the asymptote of the corresponding line and their directions are opposite to the origin.

III. Isomorphism between the models

Two different models of a system of axioms are called isomorphs if it is possible to establish a bijection between the basic sets of the models which preserves the relations.

Since we call two configurations of a model congruent if one of the configurations can be transformed into the other one by finite number of reflections in a line, in this paper *we call two different models of the hyperbolic plane isomorphs if there exist a bijection that preserves points, lines, incidence, order and reflection in a line.*

III.1 The isomorphism between the Weierstrass and our polar-coordinate models

Since the points, lines, incidence, order and reflection in a line in the model $\mathcal{P}^2$ were defined as the orthogonal projections of the model $\mathcal{W}^2$ onto the xy-plane, and this orthogonal projection is a bijection between the models on the base of the theorems II.1, II.2 and their corollaries, it yields an isomorphism between the models.

III.2 The isomorphism between the Weierstrass and the Cayley–Klein models

Lemma III.2.1. The central projection with the centre **O** between the models $\mathcal{W}^2$ and $\mathcal{C}^2$ (built on the plane z=1 with the centre **K**(0,0,1)) is a bijection that preserves points, lines, incidence and order.

Proof. The projecting line **OP** belonging to the point $\mathbf{P}\in \mathcal{W}^2$ goes inside the asymptotic cone A^2 of H^2. The intersection of A^2 and the plane z=1 is a unit circle k^2 with the centre **K**(0,0,1) (*Fig. 7*). The intersection point **P'** of the projecting line **OP** and the plane z=1 is an inner point of k^2, therefore a point of C^2 too. We can assign uniquely a point $\mathbf{P}\in \mathcal{W}^2$ to any point $\mathbf{P}'\in C^2$, hence the projection is a bijection and preserves points.

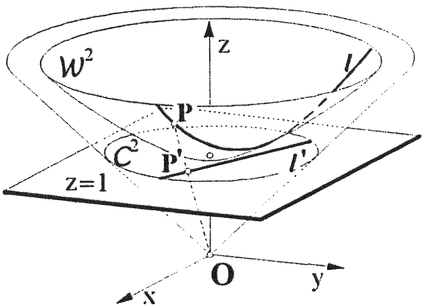

Figure 7

The plane z=1 and the projecting plane $[l,\mathbf{O}]$ belonging to line $l\subset \mathcal{W}^2$ intersect each other in line l', whose points inside the circle k^2 form line $l'\subset C^2$, therefore the correspondence preserves lines.

A point $\mathbf{P}\in \mathcal{W}^2$ lies on a line $l\subset \mathcal{W}^2$ if the projecting line **OP** lies in the projecting plane $[l,\mathbf{O}]$, namely, if the point $\mathbf{P}'\in C^2$ lies on the line $l'\subset C^2$. It means that the correspondence preserves incidence.

If we consider the collinear points **A**, **B**, $\mathbf{C}\in \mathcal{W}^2$, and **C** is between **A** and **B**, then the half line **OC** lies in the **AOB** angular domain, which means that **C'** is between **A'** and **B'**. □

Lemma III.2.2. The reflections in a line in the Weierstrass and the Cayley–Klein models correspond to each other.

Proof. Let a line $t\subset \mathcal{W}^2$ and a point $\mathbf{P}\in \mathcal{W}^2$, $\mathbf{P}\notin \boldsymbol{t}$ be given. Let $\overline{\mathbf{P}}\in \mathcal{W}^2$ be the image of **P** under the reflection in axis the $\boldsymbol{t}$. Denote by $l\subset \mathcal{W}^2$ the line passing through **P** and $\overline{\mathbf{P}}$, by e the Euclidean line passing through **P** and $\overline{\mathbf{P}}$ (*Fig. 8*). Let $\boldsymbol{t}'$, $\boldsymbol{l}'$ and **P'** be the projections of $\boldsymbol{t}$, $\boldsymbol{l}$ and **P**, respectively. Denote by **T'** the pole of $\boldsymbol{t}'$ with respect to the base circle of C^2 in the plane z=1 (in Euclidean sense **T'** can be an ideal point). Let $\overline{\mathbf{P}}'\in C^2$ be the image of **P'** under the reflection in the axis t' and **L'** the point of intersection of t' and l'. Then $(\mathbf{P}'\overline{\mathbf{P}}'\mathbf{T}'\mathbf{L}') = -1$.

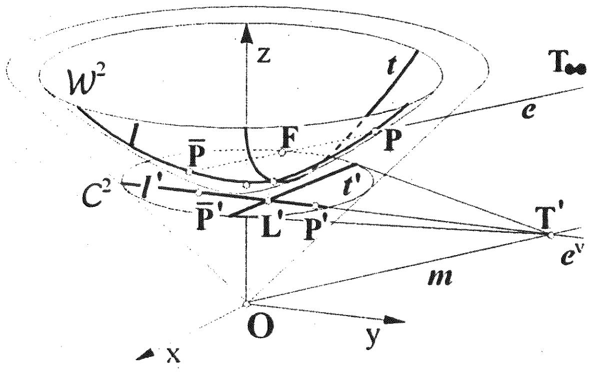

Figure 8

The line e is parallel to the direction of the planar affinity which defines the reflection in the line t in W^2, namely, it is parallel to the line of intersection m of the tangent planes belonging to $[t]$. Plane $[t]$ is the polar plane of line m with respect to the asymptotic cone – therefore with respect to the surface H^2 too. Let f be an arbitrary Euclidean line which intersects the line m, or parallel to m (in this case the point of intersection is a Euclidean ideal point). It is well-known that the points of intersection of the line f and H^2 separate harmonically the point of intersection of f and m and the point of intersection of f and $[t]$, i.e., their cross-ratio is equal to -1. Thus, if the point of intersection of e and m is $\mathbf{T}_\infty$ and the point of intersection of e and $[t]$ is $\mathbf{F}$, then $(\mathbf{P}\bar{\mathbf{P}}\mathbf{T}_\infty\mathbf{F}) = -1$. The point $\mathbf{T}'$ lies on m because the tangent lines from $\mathbf{T}'$ to the base circle of C^2 lie on the tangent planes determined by $[t]$.

We project line e from the point $\mathbf{O}$ onto the plane z=1. Let e^V be the projection of e. Then $\mathbf{T}'$ and $\mathbf{L}'$ will be the projections of $\mathbf{T}_\infty$ and $\mathbf{F}$, respectively. Since the central projection preserves cross-ratio, it follows $(\mathbf{P}\bar{\mathbf{P}}\mathbf{T}_\infty\mathbf{F}) = (\mathbf{P}'\bar{\mathbf{P}}'\mathbf{T}'\mathbf{L}') = -1$, i.e., $\bar{\mathbf{P}}'$ is the projection of $\bar{\mathbf{P}}$. □

Theorem III.2.1. The Weierstrass model and the Cayley–Klein model are isomorphs.

Proof. By the help of lemmas III.2.1 and III.2.2, and the central projection with the centre $\mathbf{O}$ the isomorphism between the model W^2 and the model C^2 defined on the plane z=1 with the centre $\mathbf{K}(0,0,1)$ is proved. □

III.3 The isomorphism between the Cayley-Klein and the Poincaré's hemisphere models

The isomorphism between the models $\mathcal{C}^2$ and $\mathcal{G}^2$ is based on an orthogonal projection. It can be found in some books ([*BOLYAI*,1987], [*HORVÁTH*,1980]).

III.4 The isomorphism between the Cayley-Klein and our polar-coordinate models

We prove the isomorphism between the Cayley–Klein and our polar-coordinate models by the help of an orthogonal projection, using the isomorphism between these models and the Weierstrass model.

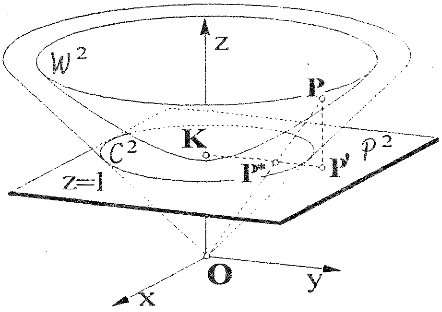

Figure 9

Theorem III.4.1. The Cayley–Klein and our polar-coordinate models are isomorphs.

Proof. If we project $\mathcal{C}^2$ from **O** onto the surface H^2 we get the $\mathcal{W}^2$. If we project perpendicularly $\mathcal{W}^2$ onto the plane z=1 we get $\mathcal{P}^2$ (*Fig. 9*). Since both projections yield isomorphisms, $\mathcal{C}^2$ and $\mathcal{P}^2$ are isomorphs too, and the product of the projections assign the models to each other bijectively. □

Corollary. By the help of the isomorphism between the models $\mathcal{C}^2$ and $\mathcal{P}^2$ we give a simple method for construction of reflection in a line in the model $\mathcal{P}^2$. This method for the construction is shown in *figure 10*.

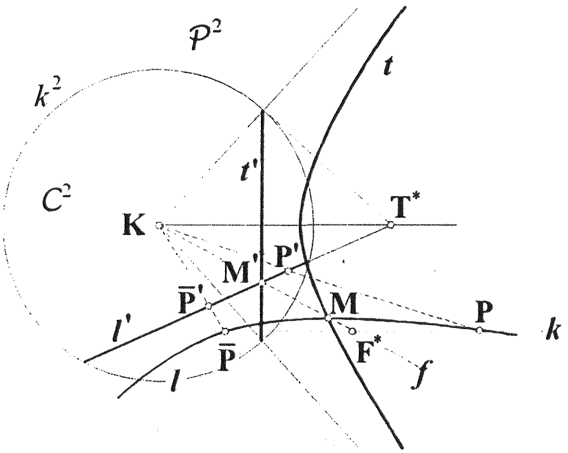

Figure 10

III.5 The isomorphism between the Poincaré's hemisphere and our polar-coordinate models

The isomorphism is based on a central projection with centre **O**.

Lemma III.5.1. Let the point $\mathbf{P} \in \mathcal{G}^2$ be given. Let $\mathbf{P}^* \in C^2$ be the orthogonal projection of **P** onto the plane z=1 and the point **P'** be the central projection of **P** from **O**. The projection of $\mathbf{P}^*$ on the surface H^2 from **O** is the point $\tilde{\mathbf{P}} \in \mathcal{W}^2$. Then the orthogonal projection of $\tilde{\mathbf{P}}$ onto the plane z=1 is the point **P'**, which belongs to the model $\mathcal{P}^2$.

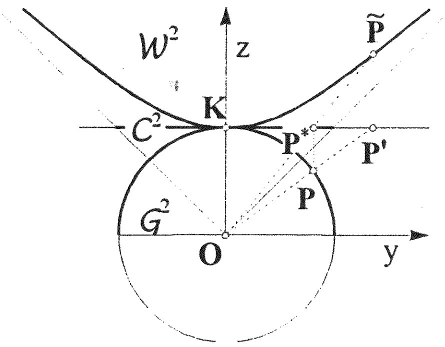

Figure 11

Proof. If **P** coincides with **K**(0,0,1) then the statement is trivial. If **P** does not coincide with **K** (*Fig. 11*), then rotate it about the z-axis into the yz-plane. Then the coordinates of **P**, **P***, $\tilde{\mathbf{P}}$ and **P'** are $(0,a,\sqrt{1-a^2})$, $(0,a,1)$, $(0,\frac{a}{\sqrt{1-a^2}},\frac{1}{\sqrt{1-a^2}})$ and $(0,\frac{a}{\sqrt{1-a^2}},1)$, respectively, that prove the correspondence of the points. □

Theorem III.5.1. The Poincaré's hemisphere and the polar-coordinate models are isomorphs.

Proof. If we project $\mathcal{G}^2$ perpendicularly onto the plane z=1 we get $\mathcal{C}^2$. If we project $\mathcal{C}^2$ from the point **O** onto H^2 we get $\mathcal{W}^2$, and if we project $\mathcal{W}^2$ perpendicularly onto the plane z=1 we get $\mathcal{P}^2$. Considering these three projections and the lemma III.5.1. we get, that the models $\mathcal{G}^2$ and $\mathcal{P}^2$ (defined in the plane z=1) are isomorphs, and the central projection with the centre **O** is a bijection between the models. □

III.6 The isomorphism between the Weierstrass and the Poincaré's hemisphere models

Lemma III.6.1. The surface H^2 is assigned to the surface G^2 under the spatial central collineation with the centre **D**(0,0,-1), the fixed plane z=1 and the counter plane xy.

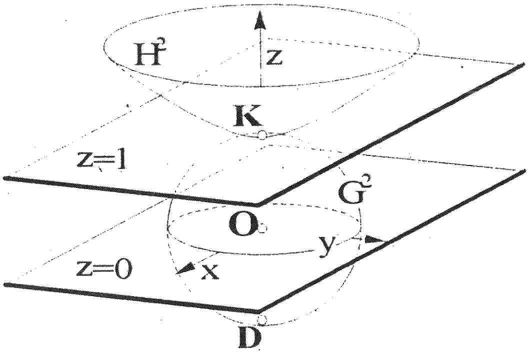

Figure 12

Proof. The spatial central collineation takes a second-order surface into a surface with the same type (in projective sense). The xy counter plane intersects the unit sphere with the centre **O**, hence the image of the sphere can be nothing else but a two-sheeted hyperboloid (*Fig. 12*). The sphere goes through the centre **D** and touches the fixed plane z=1 in the point **K**(0,0,1), therefore the hyperboloid will go through **D** and touch the plane z=1 in **K** as well. The sphere has rotational symmetry about the z-axis that is perpendicular to the fixed plane, hence the hyperboloid will have rotational symmetry about the z-axis. A projecting cone similar to A^2 with the z-axis belongs to the great circle of the sphere in the counter plane and the ideal points (in Euclidean sense) of the hyperboloid correspond to the points of the great circle of the sphere, hence the asymptotic cone of the hyperboloid is similar to A^2 with the axis z. Since the axis of revolution hyperboloid is z that since the axis of revolution of the hyperboloid is z passes through the points **K** and **D**, the centre of the hyperboloid is the midpoint **O** of the segment **KD**. The surface G^2 is that hemisphere of the unit sphere which is between the counter plane and the fixed plane, and contains **K**. Then the image of G^2 by the central collineation is the sheet of the hyperboloid containing **K**, namely, the surface H^2. □

The following theorem can be simply proved on the base of the lemma III.6.1 using a central projection with the centre **O**, similarly to the proofs of the previous isomorphisms.

Theorem III.6.1. The Poincaré's hemisphere and the Weierstrass models are isomorphs.

III.7 The isomorphism of the Klein-Poincaré model with the Poincaré's hemisphere and the Weierstrass models

By the help of a stereographic projection with the centre **D**(0,0,-1) that preserves points, circles, incidence and cross-ratio the following theorem can be simply proved.

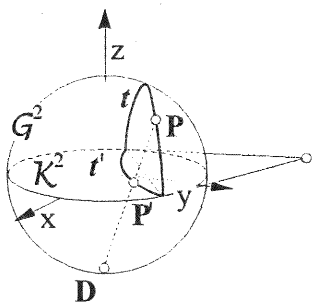

Figure 13

Theorem III.7.1. The Klein–Poincaré and the Poincaré's hemisphere models are isomorphs.

Since the central projection with the centre **D**(0,0,-1) assigns the models $\mathcal{G}^2$ and $\mathcal{W}^2$, and the models $\mathcal{G}^2$ and $\mathcal{K}^2$ to each other, this central projection ensures the isomorphism between the models $\mathcal{K}^2$ and $\mathcal{W}^2$.

Theorem III.7.2. The Klein–Poincaré and the Weierstrass models are isomorphs.

Remark. Using only one projection we can not correspond the Klein–Poincaré, the Cayley–Klein and the polar-coordinate models to each other, but by the help of the model $\mathcal{G}^2$, using two projections we can. In the 4th chapter we give the analytic connections between these models.

III.8 An organised system of the models

Figure 14 shows the previously introduced models and the connections between them.

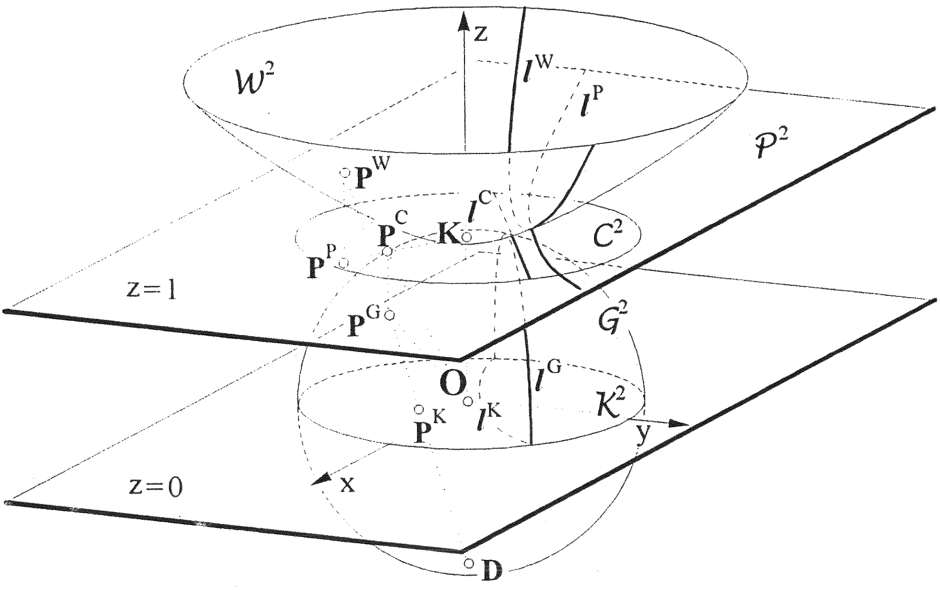

Figure 14

— If we project perpendicularly the Weierstrass model onto the plane z=1, we get our polar-coordinate model, and conversely.

— If we project with the centre **O** our polar-coordinate model onto the half (z>0) of a unit sphere, we get the Poincaré's hemisphere model, and conversely.

— If we project with the centre **O** the Weierstrass model onto the plane z=1, we get the Cayley–Klein model, and conversely.

— If we project perpendicularly the Poincaré's hemisphere model onto the plane z=1, we get the Cayley–Klein model, and conversely.

— If we project with the centre **D**(0,0,-1) the Weierstrass model onto the half (z>0) of a unit sphere, we get the Poincaré's hemisphere model, and conversely.

— If we project with the centre **D** the Poincaré's hemisphere model onto the plane z=0; we get the Klein–Poincaré model, and conversely.

— If we project with the centre **D** the Weierstrass model onto the plane z=0, we get the Klein–Poincaré model, and conversely.

IV. Metric connections

We show that the central projection with the centre **O** between the Weierstrass and the Cayley–Klein models preserves distance and angle.

Theorem. IV.1. The distance between the points **P**, $\mathbf{Q} \in \mathcal{W}^2$ is d if, and only if the distance between points **P'**, $\mathbf{Q}' \in \mathcal{C}^2$ is d.

Proof. Since the surface H^2 (in Euclidean sense) has a rotational symmetry about the z-axis we can rotate it about z so, that the plane determined by the line $l \subset \mathcal{W}^2$ (l passes through $\mathbf{P}(p_x,p_y,p_z)$ and $\mathbf{Q}(q_x,q_y,q_z)$) would be perpendicular to the plane yz. Then the coordinates of the points $\mathbf{P}'(p'_x,p'_y,1) \in \mathcal{C}^2$ and $\mathbf{Q}'(q'_x,q'_y,1) \in \mathcal{C}^2$ are $(\frac{p_x}{p_z},\frac{p_y}{p_z},1)$ and $(\frac{q_x}{q_z},\frac{q_y}{q_z},1)$, respectively (*Fig. 15*).

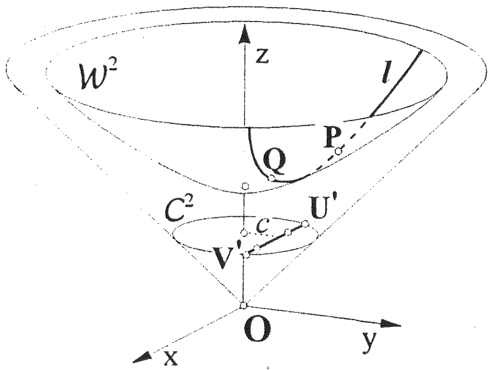

Figure 15

We defined the distance d between the points **P** and **Q** by the expression $\cosh\frac{d}{k} = -\frac{\langle \mathbf{P},\mathbf{Q}\rangle}{k^2}$.

In our case:

$$cosh\frac{d}{k} = -\frac{\langle \mathbf{P},\mathbf{Q}\rangle}{k^2} = -\frac{p_x q_x + p_y q_y - p_z q_z}{\sqrt{(p_x^2+p_y^2-p_z^2)(q_x^2+q_y^2-q_z^2)}} = \frac{1-p'_x q'_x - p'_y q'_y}{\sqrt{(1-p'^2_x-p'^2_y)(1-q'^2_x-q'^2_y)}}.$$

The line $l' \subset C^2$ is parallel to the x-axis, therefore, we define the y coordinate of each point of l' by a constant c. Then the coordinates of **P**', **Q**', **U**' and **V**' are $(p,c,1)$, $(q,c,1)$, $(-\sqrt{1-c^2},c,1)$ and $(\sqrt{1-c^2},c,1)$, respectively.

Now the previous expression of distance is:

$$cosh\frac{d}{k} = \frac{1-pq-c^2}{\sqrt{(1-p^2-c^2)(1-q^2-c^2)}}.$$

Distance ρ between points **P**' and **Q**' in the Cayley–Klein model is:

$\rho = \frac{1}{2}|k\, ln\lambda|$, where

$$\lambda = (\mathbf{P'Q'U'V'}) = \frac{-\sqrt{1-c^2}-p}{-\sqrt{1-c^2}-q} : \frac{\sqrt{1-c^2}-p}{\sqrt{1-c^2}-q} = \frac{\left(\sqrt{1-c^2}+p\right)\left(\sqrt{1-c^2}-q\right)}{\left(\sqrt{1-c^2}+q\right)\left(\sqrt{1-c^2}-p\right)}.$$

By applying algebraic transformations we get:

$$cosh\frac{\rho}{k} = cosh\frac{\frac{1}{2}|k\, ln\lambda|}{k} = \frac{\lambda+1}{2\sqrt{\lambda}} = \frac{1-pq-c^2}{\sqrt{(1-p^2-c^2)(1-q^2-c^2)}} = cosh\frac{d}{k},$$

which was to be proved. □

Theorem IV.2. The angle of lines l, $m \subset W^2$ is φ, if, and only if the angle of the lines l', $m' \subset C^2$ is φ.

Proof. Let **P** be the point of intersection of the lines l and m. We examine first the case when the point **P** lies on the z-axis.

Let the vectors $\underline{u}$ and $\underline{v}$ be the tangent vectors of the lines l and m at the point **P**. We denote by $\underline{u}'$ and $\underline{v}'$ the tangent vectors of the lines l' and m' at the point **P**'. (l' and $\underline{u}'$, as well as m' and $\underline{v}'$ lie on the same lines.) Then $\underline{u}$ and $\underline{v}$ are parallel to the xy-plane, hence the z coordinates of these vectors are zero. Thus the h-inner product of $\underline{u}$ and $\underline{v}$ is equal to their Euclidean inner-product, in addition the tangent vectors are parallel to their projections (*Fig. 16*). Hence we have:

$$\cos\varphi = \frac{\langle \underline{u},\underline{v}\rangle}{\sqrt{\langle \underline{u},\underline{u}\rangle\langle \underline{v},\underline{v}\rangle}} = \frac{(\underline{u},\underline{v})}{\sqrt{(\underline{u},\underline{u})\,(\underline{v},\underline{v})}} = \frac{(\underline{u}',\underline{v}')}{\sqrt{(\underline{u}',\underline{u}')(\underline{v}',\underline{v}')}}$$

for the angle of the lines $\boldsymbol{l}$ and $\boldsymbol{m}$.

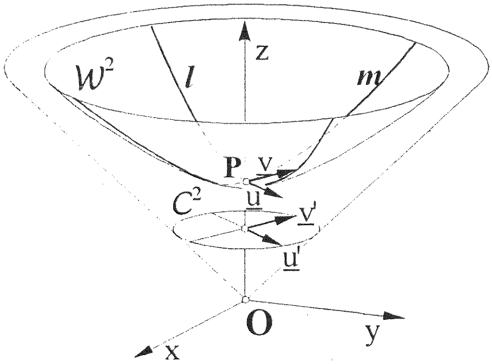

Figure 16

Since the measure of angles whose vertices are at the centre of the base circle is Euclidean in the Cayley–Klein model, it is $\cos\varphi' = \dfrac{(\underline{u}',\underline{v}')}{\sqrt{(\underline{u}',\underline{u}')(\underline{v}',\underline{v}')}}$ for the angle φ' of the lines $\boldsymbol{l}'$ and $\boldsymbol{m}'$.

Thus the equality $\varphi=\varphi'$ is proved in the above simple case.

Now we examine the case when the point of intersection **P** does not lie on the z-axis.

We rotate the surface H^2 about z so, that the point **P** would lie in the yz-plane. Let $\langle \mathbf{X},\underline{l}\rangle=0$ be the equation of the line $\boldsymbol{l}$ and $\langle \mathbf{X},\underline{m}\rangle=0$ be the equation of the line $\boldsymbol{m}$. We consider now the line $\boldsymbol{g}$ that passes through **P** and the plane determined by it is the yz-plane ($\boldsymbol{g}$ may coincide with $\boldsymbol{l}$, or $\boldsymbol{m}$). Thus the equation of $\boldsymbol{g}$ is $\langle \mathbf{X},\underline{g}\rangle=0$, where $\underline{g}(1,0,0)$ and $\boldsymbol{g}'$ is parallel to the y-axis.

We introduce a rectangular coordinate system with the centre (0,0,1) and the (ξ, η)-axes in the plane z=1. Its η-axis coincides with the line $\boldsymbol{g}'$ (Fig. 17). In this coordinate system the equation of $\boldsymbol{g}'$ is $\xi=0$, the equation of $\boldsymbol{l}'$ is $l_x\xi + l_y\eta = l_z$, and the equation of $\boldsymbol{m}'$ is $m_x\xi + m_y\eta = m_z$.

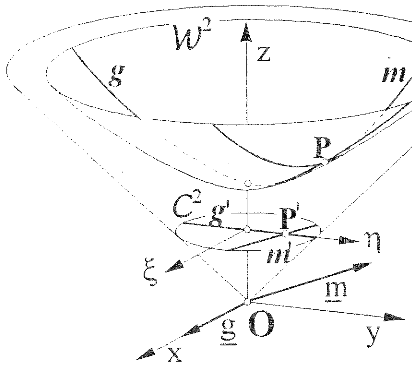

Figure 17

Suppose that the lines m and g do not coincide. First we examine the angle between m and g, and the angle between their projections. (If these two lines coincide, then we shall examine the angle between l and g, and the angle between their projections.)

We can determine the angle between the lines m and g by using their h-normal vectors as well. Thus the following expression will determine the angle φ_1 of the lines m and g: $\cos^2\varphi_1 = \dfrac{m_x^2}{m_x^2+m_y^2-m_z^2}$.

The point $\mathbf{P}'$ is the point of intersection of m' and the axis $\xi=0$, so its coordinates are $(0,\dfrac{m_z}{m_y})$, where $m_y\neq0$ and $m_z\neq0$, since $\mathbf{P}$ exists and does not lie on the z-axis.

The equation of k^2 is $\xi^2+\eta^2=1$, the equation of the polar of $\mathbf{P}'$ with respect to k^2 is $\eta=\dfrac{m_y}{m_z}$. The polar of $\mathbf{P}'$ intersects the circle k^2 at the points

$\mathbf{E}_1(\sqrt{1-\left(\dfrac{m_y}{m_z}\right)^2},\dfrac{m_y}{m_z})$ and $\mathbf{E}_2(-\sqrt{1-\left(\dfrac{m_y}{m_z}\right)^2},\dfrac{m_y}{m_z})$. The point of intersection of the polar and m' is $\mathbf{M}(\dfrac{m_z^2-m_y^2}{m_z m_x},\dfrac{m_y}{m_z})$, and the point of intersection of the polar and g' is $\mathbf{G}(0,\dfrac{m_y}{m_z})$.

The angle φ_1' of m' and g' is $\dfrac{i}{2}\ln\lambda$ in the Cayley–Klein model, where

$$\lambda = (m'g'e_1e_2) = (\mathbf{MGE_1E_2}) = \frac{\sqrt{1-\left(\frac{m_y}{m_z}\right)^2 + \frac{m_z^2 - m_y^2}{m_z m_x}}}{\sqrt{1-\left(\frac{m_y}{m_z}\right)^2 - \frac{m_z^2 - m_y^2}{m_z m_x}}}.$$

Substituting the angle $\varphi_1' = \frac{i}{2} ln\lambda$ into the expression $cos^2\varphi_1'$ and by using algebraic transformations we obtain:

$$cos^2\varphi_1' = cos^2(\frac{i}{2} ln\lambda) = \frac{(\lambda+1)^2}{4\lambda} = \frac{m_x^2}{m_x^2 + m_y^2 - m_z^2}.$$

Thus the angles φ_1 and φ_1' are the same. We proved that the angle between m and g is φ_1 if, and only if the angle of m' and g' is φ_1'.

A similar method can be used to prove the theorem for the angle φ_2 between lines l and g. Consequently, the theorem is true for the sum of the angles as well. □

We give the analytic connection between the points of the Cayley–Klein and our polar-coordinate models by the help of the isomorphism of these models and the Weierstrass model.

Theorem IV.3. Using the mapping $t \mapsto \frac{t}{\sqrt{1-t^2}}$, $t \in [0,1)$ we can construct a bijection between the points of the Cayley–Klein and the polar-coordinate models. If the coordinates of a point $\mathbf{P}^* \in \mathcal{C}^2$ are $(0,t,1)$, then the coordinates of its corresponding point $\mathbf{P}' \in \mathcal{P}^2$ are $(0, \frac{t}{\sqrt{1-t^2}}, 1)$.

Proof. Let the projection of $\mathbf{P} \in \mathcal{W}^2$ from $\mathbf{O}$ onto the plane z=1 be $\mathbf{P}^* \in \mathcal{C}^2$, the orthogonal projection of $\mathbf{P}$ onto the plane z=1 be $\mathbf{P}' \in \mathcal{P}^2$. Since the Cayley–Klein and the polar–coordinate models are isomorphs to the Weierstrass model, by the help of the Weierstrass model using the above projections we can correspond a point $\mathbf{P}^* \in \mathcal{C}^2$ to each point $\mathbf{P}' \in \mathcal{P}^2$ bijectively.

Since H^2 has a rotational symmetry, we rotate it about z so that **P** would lie on the positive quarter of the yz-plane (Fig. 9). Then the coordinates of **P** are $(0,p_y,p_z)$, $p_y \geq 0$ and $p_z=\sqrt{1+p_y^2}$. Further, $\mathbf{P}^*(0,\frac{p_y}{\sqrt{1+p_y^2}},1)$, $\mathbf{P}'(0,p_y,1)$.

Let $t = \frac{p_y}{\sqrt{1+p_y^2}}$ $(0 \leq t < 1)$ be, then we have $p_y = \frac{t}{\sqrt{1-t^2}}$. If the distance between the points $\mathbf{P}^*$ and $\mathbf{K}(0,0,1)$ is t, then the distance between $\mathbf{P}'$ and $\mathbf{K}$ is $\frac{t}{\sqrt{1-t^2}}$, namely, if the coordinates of $\mathbf{P}^*$ are $(0,t,1)$ then the coordinates of $\mathbf{P}'$ are $(0, \frac{t}{\sqrt{1-t^2}},1)$. The points **K**, $\mathbf{P}^*$ and $\mathbf{P}'$ are collinear. Further, $\mathbf{P}^*$ is always between the points **K** and $\mathbf{P}'$ (in Euclidean sense), or they coincide, because $t \leq \frac{t}{\sqrt{1-t^2}}$, if $t \in [0,1)$.

The above defined function $t \mapsto \frac{t}{\sqrt{1-t^2}}$, $t \in [0,1)$ is a one-to-one correspondence between the points of the models. □

We can give analytically the correspondence between the Klein–Poincaré and our polar-coordinate models, the Klein–Poincaré and the Cayley–Klein models by the help of the Weierstrass model similarly to the theorem IV.3. The distance between the points corresponding to each other in the hyperbolic plane ($\mathbb{H}^2$) and in their models $\mathcal{P}^2$, $\mathcal{C}^2$ and $\mathcal{K}^2$, and the centres of the models can be simply calculated – similarly to the theorem IV.3. The result can be seen in *Table 1.*

Table I

$\mathbb{H}^2$	$\mathcal{P}^2$	$\mathcal{C}^2$	$\mathcal{K}^2$
d	$sh\,d$	$th\,d$	$th\dfrac{d}{2}$
$arsh\,d$	d	$\dfrac{d}{\sqrt{1+d^2}}$	$\dfrac{d}{1+\sqrt{1+d^2}}$
$arth\,d$	$\dfrac{d}{\sqrt{1-d^2}}$	d	$\dfrac{d}{1+\sqrt{1-d^2}}$
$2arth\,d$	$\dfrac{2d}{1-d^2}$	$\dfrac{2d}{1+d^2}$	d

The domains and the ranges of the above functions are $[0,\infty)$ or $[0,1)$ depending on the models. We would be able to determine these functions in other forms as well, e.g. using only the hyperbolic functions.

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KAPCSOLATOK A HIPERBOLIKUS SÍK KÜLÖNBÖZŐ MODELLJEI KÖZÖTT

A hiperbolikus sík néhány jól ismert modelljének (Cayley-Klein, Klein-Poincaré-féle körmodellek; Poincaré-féle félgömbmodell) rövid összefoglalás után egy kevésbé ismert modelljét, a Weierstrass-féle modellt mutatjuk be. Ez a modell egy kétköpenyű forgáshiperboloid egyik köpenyén van kiépítve. A továbbiakban egy új polárkoordinátás modellnek a szintetikus felépítését mutatjuk be, amelynek definiálása a Weierstrass-féle modell merőleges vetítésével egyszerűen adódik (*3. ábra*). E modellben az alapelemeket és alaprelációkat, a távolság és szög értelmezését, a ciklusokat egyszerűen meg tudjuk adni a vetítés segítségével. A modell előnye, hogy a teljes euklideszi síkon van kiépítve. A vetítés segítségével a polárkkoordinátás modell bemutatása lényegesen egyszerűbben történik, mint *WIEGAND* cikkében (*WIEGAND*, 1992). A következő részben az ismertetett modellek egy egységes rendszerét adjuk meg (*14. ábra*), melyben merőleges, illetve centrális vetítések segítségével a köztük lévő izomorfiákat is belátjuk. Az egybevágóság megfeleltetésének alapja a tengelyes tükrözés. Az utolsó fejezetben a modellek között néhány metrikus kapcsolatra adunk bizonyítást.