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Tartalomjegyzék

THE AMAZING WORLD OF FIBONACCI SEQUENCE

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Abstract
This paper recalls a few historical facts and the definition of Fibonacci sequence, furthermore shows some recent results on Diophantine equations where the sequence appears.

1. Introduction
Fibonacci sequence is one of the most known sequences all over the world. A huge amount of information is available in mathematical books and journals on Fibonacci numbers. The sequence and the positive zero of its characteristic polynomial $x^2 - x - 1$ (the golden ratio) appear in biology, physics, chemistry, geosciences, astronomy, engineering, poetry, music, and of course in different chapters of mathematics. There exist several generalizations and extensions of the sequence, the theory of linear recurrences deals with this area. The *Fibonacci Association* (https://www.mathstat.dal.ca/fibonacci/), and the journal The *Fibonacci Quarterly* [5] have been making a big effort to popularize the sequence and gather people interested in it. For the sake of curiosity we note that there existed a rock band, *The Fibonaccis* what was formed in 1981 in Los Angeles.

The history of Fibonacci sequence goes back to Indian mathematics of 5th century BC. At the beginning of the 13th century, the Italian mathematician Fibonacci (1170-1250?) asked the following question. Suppose we have a pair of early born rabbits, and after maturing they beget every month a new pair of rabbits that becomes productive at the age of two months. Assuming that the rabbits never die, how many pairs of rabbits are there in the nth months?

Fibonacci, who was also known as Leonardo da Pisa, traveled widely with his father in North Africa and Arabia, and in his book *Liber abbaci* [10] he summarized the arithmetic and algebraic knowledge of that era. *Liber abbaci* played a basic role in the spread of the Hindu-Arabic placevalued decimal system in Europe, and among others, dealt with the rule of divisibility by 3 and the Chinese Remainder Theorem.

The Fibonacci sequence is defined by the initial values $F_0 = 0$ and $F_1 = 1$, and by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \qquad (n \geq 2).$$

Jacques Philippe Marie Binet [3] gave the explicit formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \quad n \geq 0. \tag{1}$$

(Reputedly, this formula was discovered by Abraham de Moivre in 1718 and proved ten years later by Nicolas Bernoulli.)

Although much is known about this sequence, several open problems have remained to be solved. For example, it is not known whether the Fibonacci sequence contains infinitely many primes.

In this paper, we study some scientific results of the author and his coauthors connected to Fibonacci numbers. We restrict ourselves to describe the mathematical content, but not the proofs of the statements.

Finally, we present a quote of ALBERT EINSTEIN: "[The golden proportion] is a scale of proportions which makes the bad difficult [to produce] and the good easy."

2. Some number theoretical properties

2.1. Diophantine triples

A Diophantine *m*-tuple is a set of $\{a_1, \ldots, a_m\}$ of positive rational numbers or integers such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$. DIOPHANTUS found the rational quadruple {1/16, 33/16, 17/4, 105/16}, while PIERRE FERMAT found the first integer quadruple {1, 3, 8, 120}. Infinitely many Diophantine quadruples of integers are known. Very recently HE, TOGBÉ and ZIEGLER [6] showed the non-existence of Diophantine quintuples, and closed a long standing challenging mathematical conjecture.

Now the following variant of this problem is investigated: how many triples of distinct positive integers $\{a, b, c\}$ exist such that $ab+1$, $ac+1$, $bc+1$ are all three Fibonacci numbers. The main result here is that in fact there are no such triples.

Theorem 1. (LUCA, F. – SZALAY L., [8]) *There do not exist positive integers $a < b < c$ such that*

$$ab + 1 = F_x, \qquad ac + 1 = F_y, \qquad bc + 1 = F_z, \tag{2}$$

where $x < y < z$ are positive integers.

2.2. Balancing with Fibonacci powers

Here we investigate the Diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + F_{n+2}^\ell + \cdots + F_{n+r}^\ell \tag{3}$$

in positive integers n, r, k, ℓ with $n \geq 2$. We stated the following conjecture.

Conjecture 1. ([2]) *The only quadruple $(n, r, k, \ell) = (4, 3, 8, 2)$ of positive integers satisfy equation (3).*

The conjecture was completely proved later by ALVARADO, DUJELLA and LUCA [1].

Theorem 2. (BEHERA A. – LIPTAI K. – PANDA G. K. – SZALAY L., [2]) *The Diophantine equation*

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + F_{n+2}^\ell + \cdots + F_{n+r}^\ell$$

has no solution in positive integers $n \geq 2$ and r if

- $k \leq \ell$,
- $k = 2,3$, and $\ell = 1$,
- $k = 3$, and $\ell = 2$.

2.3 Modified Brocard-Ramanujan problem

BROCARD [4] (in 1876), and later (in 1913) independently RAMANUJAN [11] posed the problem of determining all integer solutions of the Diophantine equation

$$n! + 1 = m^2.$$

It is called Brocard-Ramanujan Diophantine equation, and the only known solutions are

$$(n, m) = (4, 5); (5, 11); (7, 71).$$

The variant

$$F_n F_{n+1} \cdots F_{n+k-1} + 1 = F_m^2 \tag{4}$$

of the unsolved Brocard-Ramanujan problem was investigated by MARQUES [9]. Applying Primitive Divisor Theorem, the author proved that (4) has no solution in positive integers n, k and m.

Consider now the following generalization. Replace the Fibonacci sequence by any binary recurrence $\{G_n\}_{n=0}^{\infty}$, and suppose that the subscripts of the terms in the product on left hand side of (4) do not necessarily form an arithmetic progression with difference 1. More precisely, we will examine the Diophantine equation

$$G_{n_1} G_{n_2} \cdots G_{n_k} + 1 = G_m^2 \tag{5}$$

in integers $k \geq 1$, $m \geq 0$ and $0 \leq n_1 < n_2 < \cdots < n_k$.

In [12], the complete solution to (5) is provided if the terms G_n are either the Fibonacci numbers, or the Lucas numbers, or they satisfy the relation $G_n = AG_{n-1} - G_{n-2}$, where A is any positive integer and $G_0 = 0$, $G_1 = 1$.

Putting

$$\varepsilon = \varepsilon(m) = \begin{cases} 2, & \text{if } m \text{ is even;} \\ 1, & \text{if } m \text{ is odd,} \end{cases}$$

we can formulate the result associated to the Fibonacci sequence.

Theorem 3. (SZALAY, L., [12]). *The Diophantine equation*

$$F_{n_1} F_{n_2} \cdots F_{n_k} + 1 = F_m^2 \tag{6}$$

in positive integers k, m and $3 \leq n_1 < n_2 < \cdots < n_k$ has an infinite family of solutions given by

$$F_{m-\varepsilon} F_{m+\varepsilon} + 1 = F_m^2, \qquad m \geq 5.$$

Beside there exist only two sporadic solutions: $F_4 + 1 = F_3^2$ *and* $F_6 + 1 = F_4^2$.

2.4. On the solution of Pell equations

Let $d > 1$ be a positive integer which is not a perfect square. Consider the Pell equation

$$X^2 - dY^2 = \pm 1. \tag{7}$$

All its positive integer solutions (X, Y) are given by

$$X_n + Y_n\sqrt{d} = (X_1 + Y_1\sqrt{d})^n$$

for some positive integer n, where (X_1, Y_1) is the smallest positive solution. In several recent papers, the following problem was investigated. Let $\mathbf{U} = \{U_n\}_{n \geq 0}$ be some interesting sequence of positive integers. What can one say about the square-free integers d such that the equation $X_n \in \mathbf{U}$ has at least two solutions n? For most sequences, one expects that the answer to such a question would be that the equation $X_n \in \mathbf{U}$ has at most one positive integer solution n for any given d except maybe for a few (finitely many) values of d.

Now let **U** = $\{F_m F_\ell : m \geq \ell \geq 1\}$ be the sequence of products of two Fibonacci numbers. The first few members of **U** are

$$\mathbf{U} = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 15, 16, 21, 24, 25, 26, 34, 39, 42, \ldots\}.$$

Our result is the following.

Theorem 4. (KAFLE, B. – LUCA, F. – MONEJANO, A. – SZALAY, L. – TOGBÉ, A., [7]) *For each square-free integer $d \geq 2$ there is at most one n such that*

$$X_n = F_\ell F_m,$$

except for $d = 2, 3, 5$ for which

- $X_1 = 1$, $X_2 = 3$ $(d = 2)$,
- $X_1 = 2$, $X_2 = 9$ $(d = 5)$,
- $X_1 = 2$, $X_2 = 26$ $(d = 3)$.

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