

RESOLUTION OF THE EQUATION

$$(3^{x_1} - 1)(3^{x_2} - 1) = (5^{y_1} - 1)(5^{y_2} - 1)$$

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ABSTRACT. Consider the diophantine equation $(3^{x_1} - 1)(3^{x_2} - 1) = (5^{y_1} - 1)(5^{y_2} - 1)$ in positive integers $x_1 \leq x_2$, and $y_1 \leq y_2$. Each side of the equation is a product of two terms of a given binary recurrence, respectively. In this paper, we prove that the only solution to the title equation is $(x_1, x_2, y_1, y_2) = (1, 2, 1, 1)$. The main novelty of our result is that we allow products of two terms on both sides.

1. INTRODUCTION

This paper is devoted to investigation of positive integers having a specific but analogous structure of digits in two distinct integer bases. We worked out the details only for the bases 3 and 5, but our approach and arguments should work for other bases, as well. More precisely, we determine the solutions to the diophantine equation

$$(3^{x_1} - 1)(3^{x_2} - 1) = (5^{y_1} - 1)(5^{y_2} - 1) \quad (1.1)$$

in positive integers $x_1 \leq x_2$ and $y_1 \leq y_2$.

Senge and Strauss [10] proved that the number of integers for which the sum of digits simultaneously in base a and b do not exceed a given bound is finite if and only if $(\log a)/(\log b)$ not rational. Their method is not effective, and it motivated Stewart [8] to exhibit a lower bound for the sum of the digits in base a and b . To be precise he proved the following theorem. Assume that $a, b, n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha, \beta \in \mathbb{N}$ with $\alpha < a$ and $\beta < b$. If $N(\alpha, a)$ denotes the number of digits different from α in the canonical expansion of n in base a

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$(N(\beta, b)$ analogously), then

$$N(\alpha, a) + N(\beta, b) > \frac{\log \log n}{\log \log \log n + C} - 1, \quad (n > 25)$$

provided by $(\log a)/(\log b)$ is irrational. Here C is an effectively computable positive real number depending on a and b only.

This result is followed by several papers on Diophantine equations concerning multi-base representation of integers, see, for example [2] and the references therein.

Both sides of equation (1.1) can also be considered as the product of two terms of a binary recurrence, respectively. More generally, but only with one term on both sides Schlickewei and Schmidt [9] characterized all the pairs of recurrences (G, H) having infinitely many solution to $G_x = F_y$. Ddamulira, Luca and Rakotomalala [5] considered first the akin problem with two terms on one side. They gave all Fibonacci numbers which are products of two Pell numbers, and all Pell numbers which are products of two Fibonacci numbers. Hence the equations $F_z = P_x P_y$, and $P_z = F_x F_y$ were completely solved. In this paper, we allow products with two terms on both sides, where the two binary recurrences are representatives from the same class of sequences, which is a novel feature. The technique used in our proof is a variant of the combination of Baker's method and reduction procedures like LLL-algorithm, and a generalization of a result of Baker and Davenport by Dujella and Pethő [6]. We mention that a similar approach should work for equations of the same type involving products of more terms. However, this will certainly increase the amount of necessary computations. The principal result is recorded in the following

Theorem 1.1. *If equation (1.1) holds for the positive integers $x_1 \leq x_2$ and $y_1 \leq y_2$, then*

$$x_1 = 1, x_2 = 2, y_1 = 1, y_2 = 1.$$

We note that the method of Bertók and Hajdu described in [1] probably also helps to solve (1.1). This was confirmed by Bertók via a personal communication.

2. PRELIMINARIES

Here we list a few results which will be necessary later. Put $\lambda = \log 5 / \log 3$.

Lemma 2.1. *If $a \geq 3$ is a real number and x_1, x_2 are positive integers, then*

$$a^{x_1+x_2-1} < (a^{x_1} - 1)(a^{x_2} - 1) < a^{x_1+x_2}.$$

Proof. The second inequality is obvious. The first one follows from

$$1 - \frac{1}{a^{x_1}} - \frac{1}{a^{x_2}} + \frac{1}{a^{x_1+x_2}} > 1 - \frac{1}{a^{x_1}} - \frac{1}{a^{x_2}} \geq 1 - \frac{1}{3^{x_1}} - \frac{1}{3^{x_2}} \geq \frac{1}{3} \geq \frac{1}{a}.$$

□

Corollary 2.0.1. *Assume that the positive integers x_1, x_2, y_1 and y_2 satisfy (1.1). Then*

- $(x_1 + x_2 - 1) \log 3 < (y_1 + y_2) \log 5$,
- $(y_1 + y_2 - 1) \log 5 < (x_1 + x_2) \log 3$.

Proof. Apply Lemma 2.1 and (1.1). \square

Lemma 2.2. *If $x_1 + x_2 \leq 3$ or $y_1 + y_2 \leq 3$ holds with the positive integers $x_1 \leq x_2$ and $y_1 \leq y_2$, then (1.1) possesses only the solution $(x_1, x_2, y_1, y_2) = (1, 2, 1, 1)$.*

Proof. The statement easily follows by directly checking all four possible cases. \square

Lemma 2.3. *Assume that (1.1) holds, moreover $y_1 + y_2 \geq 4$. Then $y_1 + y_2 < x_1 + x_2$.*

Proof. A short calculation admits $y_1 + y_2 < \lambda(y_1 + y_2 - 1)$. Now the second statement of Corollary 2.0.1 proves the lemma. \square

Lemma 2.4. *Equation (1.1) implies $2 \nmid y_i$, $4 \nmid x_i$, where $i \in \{1, 2\}$.*

Proof. Consider (1.1) modulo 3, and 5, respectively. \square

Lemma 2.5. *If the real numbers x and K satisfy $|e^x - 1| < K < 3/4$, then $|x| < 2K$.*

Proof. The assertion can be easily checked. \square

We need the following theorem from the theory of lower bounds on linear forms in logarithms of algebraic numbers. Recall Theorem 9.4 of [3], which is a modified version of a result of Matveev [7]. Let \mathbb{L} be an algebraic number field of degree $d_{\mathbb{L}}$ and let $\eta_1, \eta_2, \dots, \eta_l \in \mathbb{L}$ not 0 or 1 and d_1, \dots, d_l be nonzero integers. Put

$$D = \max\{|d_1|, \dots, |d_l|, 3\} \quad \text{and} \quad \Gamma = \prod_{i=1}^l \eta_i^{d_i} - 1.$$

Let A_1, \dots, A_l be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for } j = 1, \dots, l,$$

where for an algebraic number η with minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(u)}) \in \mathbb{Z}[X]$$

with positive a_0 , we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{u} \left(\log a_0 + \sum_{j=1}^u \max\{0, \log |\eta^{(j)}|\} \right).$$

Lemma 2.6. *If $\Gamma \neq 0$ and $L \subseteq R$, then*

$$\log |\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_L^2 (1 + \log d_L)(1 + \log D) A_1 A_2 \cdots A_l.$$

We also refer to the Baker-Davenport reduction method (see [6, Lemma 5a]), which will be useful to reduce the bounds arising at the application of Lemma 2.6.

Lemma 2.7. *Let $\kappa \neq 0$ and μ be real numbers. Assume that M is a positive integer. Let P/Q be a convergent of the continued fraction expansion of κ such that $Q > 6M$, and put*

$$\xi = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\xi > 0$, then there is no solution of the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$\frac{\log(AQ/\xi)}{\log B} \leq k \quad \text{and} \quad m \leq M.$$

3. PROOF OF THEOREM 1.1

3.1. The first bound. Recall that $\lambda = \log 5 / \log 3$. Suppose that the positive integers $x_1 \leq x_2$ and $y_1 \leq y_2$ satisfy (1.1). According to Lemma 2.2 we may assume $x_1 + x_2 \geq 4$ and $y_1 + y_2 \geq 4$. Then

$$\left| \frac{3^{x_1+x_2}}{5^{y_1+y_2}} - 1 \right| = \left| \frac{3^{x_1} + 3^{x_2} - 5^{y_1} - 5^{y_2}}{5^{y_1+y_2}} \right| < \frac{4 \max\{3^{x_2}, 5^{y_2}\}}{5^{y_1+y_2}},$$

and denote this upper bound by B_1 . The assumption $\max\{3^{x_2}, 5^{y_2}\} = 5^{y_2}$ leads immediately to $B_1 = 4/5^{y_1}$. Contrary, if $\max\{3^{x_2}, 5^{y_2}\} = 3^{x_2}$, then

$$B_1 = \frac{4 \cdot 3^{x_2}}{5^{y_1+y_2}} = \frac{4}{5^{y_1+y_2-x_2/\lambda}} < \frac{4}{5^{(x_1+x_2-1)/\lambda-x_2/\lambda}} = \frac{4 \cdot 5^{1/\lambda}}{5^{x_1/\lambda}} = \frac{12}{5^{x_1/\lambda}},$$

where the inequality follows from Corollary 2.0.1. Thus we conclude

$$\left| \frac{3^{x_1+x_2}}{5^{y_1+y_2}} - 1 \right| < \frac{12}{5^{\min\{x_1/\lambda, y_1\}}}. \quad (3.1)$$

Let the term in the absolute value of left-hand side of (3.1) be denoted by Γ_1 , which is obviously non-zero. Put $z = \max\{x_1 + x_2, y_1 + y_2\}$. Clearly, Lemma 2.3 gives $z = x_1 + x_2$. Now we apply Lemma 2.6 with $l = 2$, $\eta_1 = 3$, $\eta_2 = 5$, $L = Q$, $D = z$, $A_1 = \log 3$, $A_2 = \log 5$. This provides

$$\log |\Gamma_1| > -1.4 \cdot 10^9 Z,$$

where $Z = (1 + \log z)$, and then together with (3.1) we obtain

$$\min\{x_1/\lambda, y_1\} < 8.7 \cdot 10^8 Z.$$

For the next calculations we distinguish two cases.

Case 1. Assume $3^{x_1} < 5^{y_1}$, or equivalently $x_1/\lambda < y_1$. Equation (1) leads to

$$|\Gamma_{2,1}| := \left| \frac{(3^{x_1} - 1)3^{x_2}}{5^{y_1+y_2}} - 1 \right| = \left| \frac{3^{x_1} - 5^{y_1} - 5^{y_2}}{5^{y_1+y_2}} \right| < \frac{3^{x_1} + 2 \cdot 5^{y_2}}{5^{y_1+y_2}} = \frac{\frac{3^{x_1}}{5^{y_2}} + 2}{5^{y_1}} < \frac{3}{5^{y_1}}. \quad (3.2)$$

In order to use Lemma 2.6 again, now for $|\Gamma_{2,1}|$ we specify $l = 3$, $\eta_1 = 3^{x_1} - 1$, $\eta_2 = 3$, $\eta_3 = 5$, $L = Q$. Moreover fix $D = \max\{1, x_2, y_1 + y_2\} < z$, $A_2 = \log 3$, $A_3 = \log 5$. Finally,

$$h(3^{x_1} - 1) < (\log 3)x_1 = (\log 5)x_1/\lambda < (\log 5) \cdot 8.7 \cdot 10^8 Z < 1.5 \cdot 10^9 Z = A_1.$$

Case 2. Now let $3^{x_1} > 5^{y_1}$. Hence $x_1/\lambda > y_1$. Equation (1) implies

$$|\Gamma_{2,2}| := \left| \frac{(5^{y_1} - 1)5^{y_2}}{3^{x_1+x_2}} - 1 \right| = \left| \frac{5^{y_1} - 3^{x_1} - 3^{x_2}}{3^{x_1+x_2}} \right| < \frac{5^{y_1} + 2 \cdot 3^{x_2}}{3^{x_1+x_2}} = \frac{\frac{5^{y_1}}{3^{x_2}} + 2}{3^{x_1}} < \frac{3}{3^{x_1}}. \quad (3.3)$$

For $|\Gamma_{2,2}|$ we have $l = 3$, $\eta_1 = 5^{y_1} - 1$, $\eta_2 = 5$, $\eta_3 = 3$, $L = Q$, $D = \max\{1, y_2, x_1 + x_2\} = z$, $A_2 = \log 5$, $A_3 = \log 3$, and $h(5^{y_1} - 1) < (\log 5)y_1 < A_1$.

Observe that we are able to apply Lemma 2.6 simultaneously for $|\Gamma_{2,1}|$ and $|\Gamma_{2,2}|$ since up to the order we have the same parameters. It gives

$$\log |\Gamma_{2,i}| > -3.798 \cdot 10^{20} Z^2, \quad (i = 1, 2)$$

consequently by (3.2), and (3.3), respectively we derive

$$(\log 5)y_1 - \log 3 < (\log 5)y_1 < h_1 Z^2 \quad \text{and} \quad (\log 3)x_1 - \log 3 < (\log 3)x_1 < h_1 Z^2, \quad (3.4)$$

where $h_1 = 3.8 \cdot 10^{20}$.

Return again to the conditions of the separation of Cases 1 and 2 for a while.

Case 1. ($3^{x_1} < 5^{y_1}$, $x_1/\lambda < y_1$.) Equation (1) also leads to

$$|\Gamma_{3,1}| := \left| \frac{(3^{x_1} - 1)3^{x_2}}{(5^{y_1} - 1)5^{y_2}} - 1 \right| = \left| \frac{3^{x_1} - 5^{y_1}}{(5^{y_1} - 1)5^{y_2}} \right| < \frac{3^{x_1} + 5^{y_1}}{(5^{y_1} - 1)5^{y_2}} = \frac{\frac{3^{x_1}}{5^{y_1}} + 1}{\left(1 - \frac{1}{5^{y_1}}\right) 5^{y_2}} < \frac{3}{5^{y_2}}. \quad (3.5)$$

Preparing the application of Lemma 2.6, for $|\Gamma_{3,1}|$ we fix $l = 4$, $\eta_1 = 3^{x_1} - 1$, $\eta_2 = 3$, $\eta_3 = 5^{y_1} - 1$, $\eta_4 = 5$, $L = Q$. Furthermore $D = \max\{1, x_2, 1, y_2\} < z$, $A_1 = 1.5 \cdot 10^9 Z$, $A_2 = \log 3$, $A_4 = \log 5$, and $h(5^{y_1} - 1) < (\log 5)y_1 < 3.8 \cdot 10^{20} Z^2 = A_3$. We will see soon, that Case 2 essentially admits the same parameters.

Case 2. ($3^{x_1} > 5^{y_1}$, $x_1/\lambda > y_1$.) We derive from (1) that

$$|\Gamma_{3,2}| := \left| \frac{(5^{y_1} - 1)5^{y_2}}{(3^{x_1} - 1)3^{x_2}} - 1 \right| = \left| \frac{5^{y_1} - 3^{x_1}}{(3^{x_1} - 1)3^{x_2}} \right| < \frac{5^{y_1} + 3^{x_1}}{(3^{x_1} - 1)3^{x_2}} = \frac{\frac{5^{y_1}}{3^{x_1}} + 1}{\left(1 - \frac{1}{3^{x_1}}\right) 3^{x_2}} < \frac{3}{3^{x_2}}. \quad (3.6)$$

To bound $|\Gamma_{3,2}|$ from below let $l = 4$, $\eta_1 = 5^{y_1} - 1$, $\eta_2 = 5$, $\eta_3 = 3^{x_1} - 1$, $\eta_4 = 3$, $L = Q$, $D = \max\{1, y_1, 1, x_2\} < z$, $A_1 = 1.5 \cdot 10^9 Z$, $A_2 = \log 5$, $A_4 = \log 3$, and $h(3^{x_1} - 1) < (\log 3)x_1 < 3.8 \cdot 10^{20} Z^2 = A_3$.

Thus, Lemma 2.6 returns with

$$\log |\Gamma_{3,i}| > -1.58 \cdot 10^{43} Z^4, \quad (i = 1, 2),$$

and finally with $h_2 = 1.6 \cdot 10^{43}$ we have

$$(\log 5)y_2 - \log 3 < (\log 5)y_2 < h_2 Z^4 \quad \text{and} \quad (\log 3)x_2 - \log 3 < (\log 3)x_2 < h_2 Z^4. \quad (3.7)$$

Now we combine (3.4) and (3.7) to bound $y_1 + y_2$ and $x_1 + x_2$. Recall that $z = \max\{x_1 + x_2, y_1 + y_2\} = x_1 + x_2$, $Z = 1 + \log z$. The right-hand sides yield

$$(\log 3)(x_1 + x_2) - 2 \log 3 < h_1 Z^2 + h_2 Z^4.$$

In case of the left-hand sides together with Corollary 2.0.1 we find

$$(\log 3)(x_1 + x_2 - 1) - 2 \log 3 < (\log 5)(y_1 + y_2) - 2 \log 3 < h_1 Z^2 + h_2 Z^4.$$

Both inequalities provide upper bounds on z , the result is recorded in the following

Proposition 3.1. *Assume that $z = \max\{x_1 + x_2, y_1 + y_2\} \geq 4$. Then for the solutions of (1.1)*

$$z = x_1 + x_2 < 3 \cdot 10^{51} \quad (3.8)$$

holds.

Note that in Proposition 3.1 we have $z = x_1 + x_2$ since $x_1 + x_2 > y_1 + y_2$ is obvious from Lemma 2.3. Moreover $y_1 + y_2 < 2.1 \cdot 10^{51}$ follows from Proposition 3.1 and Corollary 2.0.1.

3.2. The second bound. In the sequel we assume $x_1 \geq 3$ and $y_1 \geq 2$. The remaining cases where $x_1 < 3$ or $y_1 < 2$ will be handled later, in Subsection 3.5.

Now $\min\{x_1/\lambda, y_1\} \geq 2$, hence the right-hand side of (3.1) does not exceed $12/25 < 3/4$. Consequently, Lemma 2.5 with $x = (x_1 + x_2) \log 3 - (y_1 + y_2) \log 5$ implies

$$|\Gamma_4| := |(x_1 + x_2) \log 3 - (y_1 + y_2) \log 5| < \frac{24}{5^{\min\{x_1/\lambda, y_1\}}}. \quad (3.9)$$

For the computational aspects of the application of LLL algorithm we refer to the book of H. Cohen [4], page 58-63, moreover the `LLL(lvect, integer)` command of the package IntegerRelation in Maple. We implemented the computations in Maple by following Cohen's approach. The LLL-algorithm (with $x_1 + x_2 < 3 \cdot 10^{51}$, $y_1 + y_2 < 2.1 \cdot 10^{51}$) provides

$$4.5 \cdot 10^{-56} < |\Gamma_4|,$$

and combining it with (3.9) we derive $\min\{x_1/\lambda, y_1\} < 81.2$. Thus we have the following

Proposition 3.2. *Assume that $x_1 \geq 3$, $y_1 \geq 2$. If (1.1) holds, then*

$$\min\{x_1/\lambda, y_1\} \leq 81.2.$$

3.3. The third bounds. Recall (3.2) and (3.3), respectively, in accordance with the two cases of Subsection 3.1.

Case 1. ($3^{x_1} < 5^{y_1}$.) Since $3/5^{y_1} < 3/4$, by Lemma 2.5 we obtain

$$\left| \frac{1}{\lambda} x_2 - (y_1 + y_2) + \frac{\log(3^{x_1} - 1)}{\log 5} \right| < \frac{6}{5^{y_1} \log 5} < 3.8 \cdot 5^{-y_1},$$

where $x_2 < 3 \cdot 10^{51}$, and $3 \leq x_1 < 82.9 \cdot \lambda < 118.96$. Apply the Baker-Davenport type reduction method described in Lemma 2.7 together with the parameters $M = 3 \cdot 10^{51}$, $A = 3.8$, $B = 5$, $m = x_2$, $\kappa = 1/\lambda$, $n = y_1 + y_2$, $\mu = \log(3^{x_1} - 1)/\log(5)$ with $3 \leq x_1 \leq 118$, $4 \nmid x_1$ (87 cases). Note that

$$Q_{113} = 49979470671933915311803624529695074923111987539096229 \approx 5 \cdot 10^{52}$$

is the first denominator exceeding $6M$. For the possible values of x_1 , in all cases we found $y_1 \leq 82$.

Case 2. ($3^{x_1} > 5^{y_1}$.) If $x_1 \geq 2$, then $3/3^{x_1} < 3/4$, and Lemma 2.5 admits

$$\left| \lambda y_2 - (x_1 + x_2) + \frac{\log(5^{y_1} - 1)}{\log 3} \right| < \frac{6}{3^{x_1} \log 3} < 5.5 \cdot 3^{-x_1}.$$

Here $y_2 < 2.1 \cdot 10^{51}$, and $2 \leq y_1 \leq 81$, y_1 is odd (40 possibility for y_1). Similarly to Case 1, we use Lemma 2.7, which leads to $x_1 \leq 115$.

We summarize the last computational results as follows.

Proposition 3.3. *Assume again that $x_1 \geq 3$, $y_1 \geq 2$. Then from equation (1.1) we conclude*

$$x_1 \leq 118 \quad \text{and} \quad y_1 \leq 81.$$

3.4. The final bounds, and verification. In (3.5), and (3.6) we found that $|\Gamma_{3,1}| < 3/5^{y_2}$, and $|\Gamma_{3,2}| < 3/3^{x_2}$, respectively. Supposing $y_2 \geq 2$, and $x_2 \geq 3$, it is obvious that both $|\Gamma_{3,i}| < 4/5$, subsequently we can apply Lemma 2.5. It gives

$$\left| (\log 3)x_2 - (\log 5)y_2 + \log \left(\frac{3^{x_1} - 1}{5^{y_1} - 1} \right) \right| < \frac{6}{5^{y_2}} \quad (3.10)$$

in Case 1. Knowing $x_2 < 3 \cdot 10^{51}$, $y_2 < 2.1 \cdot 10^{51}$ we use the LLL algorithm for each possible pair (x_1, y_1) with the bounds given in Proposition 3.3. To reduce the time of the calculations we also exploit that the conditions $4 \nmid x_1$, $2 \nmid y_1$, and $3^{x_1} < 5^{y_2}$ (Case 1) also hold. The procedure yields lower bound K_{x_1, y_1} for the left-hand side of (3.10) in each case, and comparing it with $6/5^{y_2}$ we obtain an upper bound on y_2 . The maximum of the upper bounds is 159. Finally,

$$3^{x_2} < 1 + \frac{5^{y_1 + y_2}}{3^{x_1} - 1} \leq 1 + \frac{5^{81 + 159}}{3^3 - 1}$$

gives $x_2 \leq 348$. The divisibility condition reduces it to $x_2 \leq 347$.

A similar treatment works for

$$\left| (\log 5)y_2 - (\log 3)x_2 + \log \left(\frac{5^{y_1} - 1}{3^{x_1} - 1} \right) \right| < \frac{6}{3^{x_2}}$$

in Case 2. It returns with $x_2 \leq 235$, and then $y_2 \leq 238$. Thus $y_2 \leq 237$.

Proposition 3.4. *Assume again that $x_1 \geq 3$, $y_1 \geq 2$. Equation (1.1) can hold only if*

$$x_2 \leq 347 \quad \text{and} \quad y_2 \leq 237.$$

Now a simple computer search shows that there is no solution to equation (1.1) if $3 \leq x_1 \leq 118$, $2 \leq y_1 \leq 81$, $x_1 \leq x_2 \leq 347$, and $y_1 \leq y_2 \leq 237$.

3.5. The remaining cases. This subsection is devoted to handle separately the cases $x_1 = 2$, $x_1 = 1$, and $y_1 = 1$.

Case $x_1 = 2$. Now equation (1.1) has the form

$$8 \cdot (3^{x_2} - 1) - (5^{y_1} - 1)(5^{y_2} - 1) = 0, \quad (3.11)$$

where the exponents x_2 , y_1 , and y_2 are positive integers, $x_2 \geq 2$. Let $m = 3^2 \cdot 7 \cdot 13 = 819$. Observe that the Carmichael function has the small value $\lambda(m) = 12$. Since $3^x \equiv 3^{x+6} \pmod{m}$ holds if $x \geq 2$, and $5^{12} \equiv 1 \pmod{m}$ fulfils, we checked the possibilities $2 \leq x_2 \leq 7$, $1 \leq y_1, y_2 \leq 12$ for the left-hand side of (3.11) modulo m , and it never gave 0. Hence there is no solution to (1.1) with $x_1 = 2$.

Case $x_1 = 1$. Equation (1.1) gives

$$2 \cdot (3^{x_2} - 1) = (5^{y_1} - 1)(5^{y_2} - 1), \quad (3.12)$$

the exponents x_2 , y_1 , and y_2 are positive integers. Since $4 \leq y_1 + y_2$, by Lemma 2.3 we may assume $x_2 \geq y_1 + y_2$.

What follows is very similar to the treatment of the Subsections 3.1 and 3.2 therefore here we give less details. From (3.12) we have

$$|\Gamma_6| := \left| \frac{2 \cdot 3^{x_2}}{5^{y_1+y_2}} - 1 \right| = \left| \frac{3 - 5^{y_1} - 5^{y_2}}{5^{y_1+y_2}} \right| < \frac{2 \cdot 5^{y_2}}{5^{y_1+y_2}} = \frac{2}{5^{y_1}}. \quad (3.13)$$

Apply again Lemma 2.6, clearly with $D = \max\{x_2, y_1 + y_2\} = x_2$. It provides immediately $\log |\Gamma_6| > -1.755 \cdot 10^{11}(1 + \log x_2)$, which together with (3.13) entails

$$y_1 < 1.1 \cdot 10^{11}(1 + \log x_2) =: K_1(x_2). \quad (3.14)$$

From equation (3.12) we can also conclude

$$0 < \Gamma_7 := \frac{(5^{y_1} - 1)5^{y_2}}{2 \cdot 3^{x_2}} - 1 = \frac{5^{y_1} - 3}{2 \cdot 3^{x_2}} < \frac{5^{y_1}}{2 \cdot 3^{x_2}} < \frac{5}{5^{y_2}}. \quad (3.15)$$

The last inequality is a consequence of Lemma 2.1 and (3.12):

$$5^{y_1+y_2-1} < (5^{y_1} - 1)(5^{y_2} - 1) = 2 \cdot (3^{x_2} - 1) < 2 \cdot 3^{x_2}.$$

Using the theorem of Matveev (Lemma 2.6) for Γ_7 , it returns with $\log \Gamma_7 > -3.459 \cdot 10^{24}(1 + \log x_2)^2$. Hence, by (3.15)

$$y_2 < 2.16 \cdot 10^{24}(1 + \log x_2)^2 =: K_2(x_2)$$

follows. Now

$$\frac{\log 3}{\log 5} x_2 = \frac{\log 3}{\log 5} (x_1 + x_2 - 1) < y_1 + y_2 < K_1(x_2) + K_2(x_2)$$

leads to the absolute bound

$$x_2 < 1.4 \cdot 10^{28}.$$

Clearly, (3.13) implies $|\Gamma_6| < 3/4$. Thus Lemma 2.5 yields

$$|x_2 \log 3 - (y_1 + y_2) \log 5 + \log 2| < \frac{4}{5^{y_1}}.$$

The application of the LLL-algorithm with the bound $y_1 + y_2 < x_2 < 1.4 \cdot 10^{28}$ leads to

$$y_1 \leq 93 =: K_1^*.$$

Now we repeat the treatment from (3.14), replacing $K_1(x_2)$ by K_1^* . Lemma 2.6 provides

$$y_2 < 3.4 \cdot 10^{14}(1 + \log x_2) =: K_2^*(x_2).$$

Henceforward

$$\frac{\log 3}{\log 5} x_2 < K_1^* + K_2^*(x_2),$$

and then

$$y_2 < x_2 < 1.92 \cdot 10^{16}.$$

The last step of this specific case is to exploit (3.15). Clearly, $5/5^{y_2} < 3/4$, subsequently

$$\left| y_2 \frac{\log 5}{\log 3} - x_2 + \frac{\log((5^{y_1} - 1)/2)}{\log 3} \right| < \frac{10}{5^{y_2} \log 3} < \frac{10}{5^{y_2}}.$$

We used the Baked-Davenport type reduction method (Lemma 2.7) for all the possible cases $y_1 = 3, 5, \dots, 93$ (46 values) and found $y_2 \leq 29$. Thus $y_1 \leq 29$, and a verification of (3.12) with finitely many integers on its right-hand side gives no solution to (3.12).

Case $y_1 = 1$. Now equation (3.1) returns with

$$(3^{x_1} - 1)(3^{x_2} - 1) = 4 \cdot (5^{y_2} - 1). \quad (3.16)$$

A complete analogue of the treatment of the *Case $x_1 = 1$* can be applied to solve (3.16). Here we omit the details, and declare again that no solution exists unless $y_1 = y_2 = 1$.

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