mathematics

Article

Some Properties of the Exeter Transformation

Peter Csiba [1,*] and László Németh [2,*]

[1] Department of Mathematics, J. Selye University, 945 01 Komárno, Slovakia
[2] Institute of Mathematics, University of Sopron, 9400 Sopron, Hungary
* Correspondence: csibap@ujs.sk (P.C.); nemeth.laszlo@uni-sopron.hu (L.N.)

Abstract: The Exeter point of a given triangle ABC is the center of perspective of the tangential triangle and the circummedial triangle of the given triangle. The process of the Exeter point from the centroid serves as a base for defining the Exeter transformation with respect to the triangle ABC, which maps all points of the plane. We show that a point, its image, the symmedian, and three exsymmedian points of the triangle are on the same conic. The Exeter transformation of a general line is a fourth-order curve passing through the exsymmedian points. We show that each image point can be the Exeter transformation of four different points. We aim to determine the invariant lines and points and some other properties of the transformation.

Keywords: Exeter transformation; Exeter point; barycentric coordinates

MSC: 51A05; 51N15; 51N20; 97G40

Citation: Csiba, P.; Németh, L. Some Properties of the Exeter Transformation. *Mathematics* **2021**, *9*, 721. https://doi.org/10.3390/math9070721

Academic Editor: Marian Ioan Munteanu

Received: 25 February 2021
Accepted: 24 March 2021
Published: 26 March 2021

1. Introduction

The Exeter point is one of the well-known triangle centers among the over 35,000 centers in the online Encyclopedia of Triangle Centers [1]. The Exeter point of a given triangle is defined from the centroid of the triangle by a drawing process. In this article, as a generalization of the definition of the Exeter point to the whole plane of the triangle, we define a so-called Exeter transformation with respect to a given triangle ABC. We show some properties of this transformation, we give the invariant figures, and we show that certain important points during the transformation lie on a conic.

Minevich and Morton [2] defined a similar, so-called "TCC-perspector", transformation with respect to $\triangle ABC$, and they gave a nice connection between the isogonal transformation and the "TCC-perspector". For more details and the history of the Exeter point see, ex., in [1–6].

For verifying our statements we use an analytical way with barycentric coordinates. The base triples of this barycentric coordinate system we use the vertices of a given triangle. There are many interesting articles dealing with the use of barycentric coordinates, and among them the works in [7–9] may be useful.

2. Exeter Transformation

Let ABC be a triangle and $A_tB_tC_t$ its tangential triangle.

Definition 1 (Exeter point)**.** *Let G be the centroid of a triangle ABC. Define A' to be the point (other than the polygon vertex A), where the triangle median through A meets the circumcircle of ABC, and define B' and C' similarly. Three lines—$A'A_t$, $B'B_t$, and $C'C_t$—intersect at a point E_x called the Exeter point of triangle ABC.*

Therefore, Exeter point is the perspector of the circum-medial triangle $A'B'C'$, and the tangential triangle $A_tB_tC_t$.

In Figure 1, the centroid of the triangle ABC is signed as G and the Exeter point as point E_x.

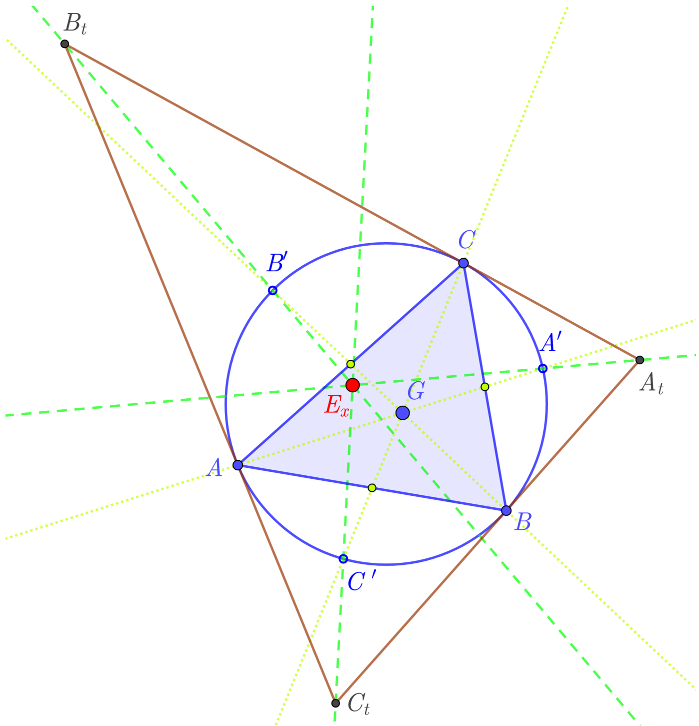

Figure 1. The Exeter point E_x.

The cevian properties of medians and its intersection (centroid) in the previous definition enable us to generalize the point construction process and obtain a new point transformation in connection to a triangle and its tangential triangle $A_tB_tC_t$:

Definition 2 (Exeter transformation)**.** *Let P be an arbitrary point in the plane of the triangle ABC. Define A' to be the point (other than the polygon vertex A) where the line AP meets the circumcircle of ABC, and define B' and C' similarly. Three lines—$A'A_t$, $B'B_t$, and $C'C_t$—intersect at a point P_e.*

The transformation by which every point P is mapped onto the point P_e by this process is called the Exeter transformation of the plane with respect to the triangle ABC (see Figure 2).

The first question naturally arises: are such three lines concurrent and we have a unique point P_e for each P? For verifying our statements, we use an analytical way with barycentric coordinates.

Let $\triangle ABC$ be the fundamental non-degenerate triangle with sidelengths $a = |BC|$, $b = |CA|$, $c = |AB|$, where the barycentric coordinates of A, B and C are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively. Let the angle C be its largest (not smaller than the others) angle. Let $\mathcal{C}$ be the circumcircle of $\triangle ABC$ with equation

$$a^2yz + b^2zx + c^2xy = 0, \tag{1}$$

and let the triangle $A_tB_tC_t$ be the tangential triangle of $\triangle ABC$. Now, the sides of $\triangle A_tB_tC_t$ are on the tangent lines to $\mathcal{C}$ at the vertices of $\triangle ABC$. Thus, if $\triangle ABC$ is an acute triangle, then the incircle of $\triangle A_tB_tC_t$ coincides with $\mathcal{C}$ (the triangle ABC is known as the Gergonne triangle of $\triangle A_tB_tC_t$). If $\triangle ABC$ is an obtuse triangle, then $\mathcal{C}$ is one of the excircles of $\triangle A_tB_tC_t$. If $\triangle ABC$ is right angled, then C_t is an ideal point. In all cases, the segment A_tB_t touches the circle $\mathcal{C}$ at point C. In the projective sense, the lines of the sides of $\triangle A_tB_tC_t$ divide the plane into four subsets. One is bounded, the others are unbounded in the affine sense. Let $\mathcal{R}$ denote the subset which contains $\mathcal{C}$.

The homogeneous barycentric coordinates of the vertices of $\triangle A_tB_tC_t$ are

$$A_t(-a^2 : b^2 : c^2), \quad B_t(a^2 : -b^2 : c^2), \quad C_t(a^2 : b^2 : -c^2).$$

Points A_t, B_t and C_t are also known as exsymmedian points with respect to $\triangle ABC$. As usual, we denote the homogeneous barycentric coordinates by $(: :)$ and the normalized (or absolute) barycentric coordinates by $(, ,)$.

We consider an arbitrary point in the plane of triangle ABC

$$P(u : v : w) = \left(\frac{u}{s}, \frac{v}{s}, \frac{w}{s}\right),$$

where $s = u + v + w \neq 0$, $P \neq A$, $P \neq B$ and $P \neq C$, so at least two coordinates are not zero ($uv \neq 0$, $uw \neq 0$ or $vw \neq 0$). Let the lines AP, BP, and CP meet $\mathcal{C}$ at A', B', and C', respectively.

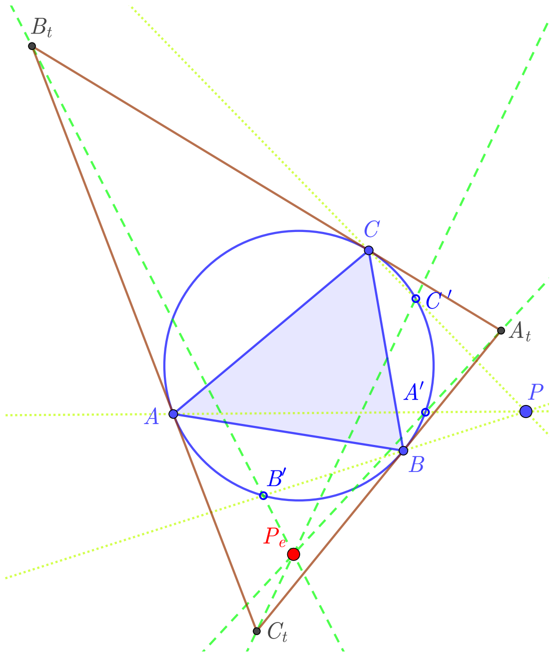

Figure 2. The Exeter transformation.

Lemma 1. *The lines A_tA', B_tB' and C_tC' from Definition 2 are concurrent for all arbitrary point P.*

Proof. The equation of the line AP is $\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ u & v & w \end{vmatrix} = wy - vz = 0$. Similarly, the equations of lines BP and CP are $wx - uz = 0$ and $vx - uy = 0$, respectively. Their intersection points with the circumcircle $\mathcal{C}$ are $A'(-a^2vw : v(b^2w + c^2v) : w(b^2w + c^2v))$, $B'(u(a^2w + c^2u) : -b^2uw : w(a^2w + c^2u))$, and $C'(u(a^2v + b^2u) : v(a^2v + b^2u) : -c^2vu)$.

The equation of line A_tA' is $\begin{vmatrix} x & y & z \\ -a^2 & b^2 & c^2 \\ -a^2vw & v(b^2w + c^2v) & w(b^2w + c^2v) \end{vmatrix} = (b^4w^2 - c^4v^2)x + a^2b^2w^2y - a^2c^2v^2z = 0$. Similarly, the equations of lines B_tB' and C_tC' are $-a^2b^2w^2x + (c^4u^2 - a^4w^2)y + b^2c^2u^2z = 0$ and $a^2c^2v^2x - b^2c^2u^2y + (a^4v^2 - b^4u^2)z = 0$, respectively. We then have from their coefficients that

$$\begin{vmatrix} b^4w^2 - c^4v^2 & a^2b^2w^2 & -a^2c^2v^2 \\ -a^2b^2w^2 & -a^4w^2 + c^4u^2 & b^2c^2u^2 \\ a^2c^2v^2 & -b^2c^2u^2 & a^4v^2 - b^4u^2 \end{vmatrix} = 0,$$

which implies the concurrency. $\square$

Let the point of concurrence of lines A_tA', B_tB', and C_tC' be P_e (Figure 2). The image of a point P under the Exeter transformation with respect to triangle ABC is the point P_e. We denote it by $ExTr(P) = P_e$. Let $G(1:1:1)$ be the centroid of $\triangle ABC$ (X_2 in [1]). Then $ExTr(G)$ is the Exeter point (X_{22} in [1]) of triangle ABC. That is why we call this transformation the Exeter transformation.

In the following, we examine the Exeter transformation and give some of its properties:

Theorem 1. *The barycentric coordinates of $P_e = ExTr(P)$, which is the image of $P(u:v:w)$ ($P \neq A$, $P \neq B$, $P \neq C$) over the Exeter transformation with respect to the triangle ABC, are*

$$\Big(a^2(-\mathfrak{p}+\mathfrak{q}+\mathfrak{r}) : b^2(\mathfrak{p}-\mathfrak{q}+\mathfrak{r}) : c^2(\mathfrak{p}+\mathfrak{q}-\mathfrak{r})\Big), \tag{2}$$

where $\mathfrak{p} = a^4v^2w^2$, $\mathfrak{q} = b^4w^2u^2$ and $\mathfrak{r} = c^4u^2v^2$.

Proof. As the lines A_tA', B_tB', and C_tC' are concurrent according to Lemma 1, in order to determine the intersection point of lines A_tA', B_tB' we solve the system of their equations, and we obtain the concurrence point P_e with barycentric coordinates (2). □

Remark 1. *If $uvw \neq 0$ (P is not on any sideline of $\triangle ABC$), then from (2) we have $P_e = ExTr(P) =$*

$$\left(a^2\left(-\frac{a^4}{u^2}+\frac{b^4}{v^2}+\frac{c^4}{w^2}\right) : b^2\left(\frac{a^4}{u^2}-\frac{b^4}{v^2}+\frac{c^4}{w^2}\right) : c^2\left(\frac{a^4}{u^2}+\frac{b^4}{v^2}-\frac{c^4}{w^2}\right)\right).$$

Remark 2. *If a triangle ABC is equilateral, so $a=b=c=1$, then the barycentric coordinates of P_e are $(u^2v^2+w^2u^2-v^2w^2 : u^2v^2-w^2u^2+v^2w^2 : -u^2v^2+u^2v^2+v^2w^2)$, and if $uvw \neq 0$, then the normalized barycentric coordinates of P_e are*

$$(1-\frac{\mathfrak{K}}{u^2}, 1-\frac{\mathfrak{K}}{v^2}, 1-\frac{\mathfrak{K}}{w^2}),$$

where

$$\mathfrak{K} = \frac{2u^2v^2w^2}{u^2v^2+u^2v^2+v^2w^2} = \frac{2}{\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2}}.$$

If we used the planar by projective coordinates (obtained from a projective base given by the points A, B, C, G where A, B, and C are triangle vertices and the centroid G is the "unit" point), instead of the barycentric with respect to the triangle ABC, we would notice that the projective coordinates are the same as barycentric in case $a=b=c=1$, but this way we would lose the Euclidean metrical properties of the Exeter transformation. However, we could extend Theorem 1 to the projective plane. Thus, the Exeter transformation works if we consider any circumconic $\mathcal{C}$ of a triangle A, B, C and its tangential triangle $A_tB_tC_t$.

Corollary 1. *The range of the Exeter transformation is $\mathcal{R}$.*

Proof. We have to prove that P_e is in $\mathcal{R}$. For this, we project the point P_e from the vertices of $\mathcal{R}$ to the sidelines of $\mathcal{R}$, namely, from the vertices A_t, B_t and C_t to the sidelines of $\triangle A_tB_tC_t$. We show that these projected points are on the sides of $\mathcal{R}$.

For example, the barycentric coordinates of each point of the line A_tB_t are $(qa^2 : -qb^2 : c^2)$, where the parameter $q \in \mathbb{R}$. If $q = \pm 1$, then we have A_t or B_t, and in the case of $q = 0$ the point coincides with C, which is one of the points of the circle $\mathcal{C}$. Thus, the parameters $q \in [-1,1]$ describe one of the segments A_tB_t, which is the side of $\mathcal{R}$. Now, we consider the equation $a^2c^2v^2x - b^2c^2u^2y + (a^4v^2 - b^4u^2)z = 0$ of the line C_tP_e (in the proof of Lemma 1, it is the line C_tC') and substitute $(qa^2 : -qb^2 : c^2)$ into the equation. After a short calculation, we express $t = -(a^4v^2 - b^4u^2)/(a^4v^4 + b^4u^2)$, where $|t| \leq 1$.

With similar calculation, we can prove that the other two projected points are on the sides of $\mathcal{R}$ as well. □

Corollary 2. *If $P = (u : v : w)$ and $P^i = (\pm u : \pm v : \pm w)$, then $ExTr(P) = ExTr(P^i)$, because (2) contains only even powers of the coordinates of P (recall $(-u : v : w) = (u : -v : -w)$). Thus, generally, there are four points which have the same image with respect to the Exeter transformation. Let $P^0 = P$, $P^1 = (-u : v : w)$, $P^2 = (u : -v : w)$, and $P^3 = (u : v : -w)$.*

Figure 3 shows the constructions of points P^i. For example, line $B'B_t$ intersects $\mathcal{C}$ in point $\overline{B}$ as well, and the intersection point of lines $B\overline{B}$ and AA' is P^1. Follow the construction of the image of P^1 with respect to the Exeter transformation, then the result is P_e. Similarly, using $\overline{A}$ and $\overline{C}$ we obtain points P^2 and P^3.

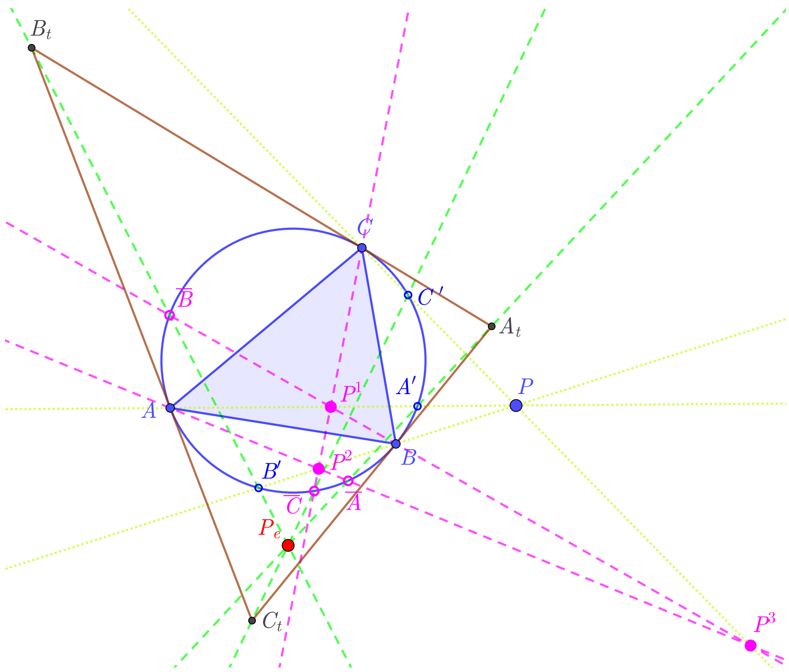

Figure 3. Four points (P, P^1, P^2, and P^3) have the same image P_e.

Corollary 3. *If we consider the centroid of $\triangle ABC$ and the vertices of the anticomplementary triangle $A_1B_1C_1$ of $\triangle ABC$ with coordinates $(1 : 1 : 1)$, $(-1 : 1 : 1)$, $(1 : -1 : 1)$ and $(1 : 1 : -1)$, respectively, then their common image is the point $(a^2(-a^4 + b^4 + c^4) : b^2(a^4 - b^4 + c^4) : c^2(a^4 + b^4 - c^4))$, which is the Exeter point (X_{22}) of $\triangle ABC$.*

Theorem 2. *The Exeter transformations of the lines AB, BC, and CA (except points A, B, C) are the points C_t, A_t and B_t, respectively.*

Proof. The equation of line AB is $z = 0$, so if P is on line AB then the coordinates are $(u, v, 0)$. Therefore, from (2), as $p = q = 0$, we have $ExTr(P) = (a^2\mathfrak{r} : b^2\mathfrak{r} : -c^2\mathfrak{r}) = (a^2 : b^2 : -c^2) = C_t$, where $\mathfrak{r} = c^4u^2v^2 \neq 0$. The proof is similar for the other lines. □

Corollary 4. *If P is on one of the lines AB, BC or CA, then some points from among P^i for $i = 0, 1, 2, 3$ coincide.*

Corollary 5. *If P, P^1, P^2, and P^3 are not on lines AB, BC, or CA, then they form a complete quadrangle with diagonal points A, B, and C.*

Theorem 3. *The Exeter transformation of a line passing through neither points A, B, and C is a fourth-order curve incident with points A_t, B_t, and C_t.*

Proof. Without lost of generalization to give a general line g we take the points T_a, T_b and T_c on lines BC, CA and AB, respectively, with baricentric coordinates $T_a(0:1:t_a)$, $T_b(t_b:0:1)$ and $T_c(1:t_c:0)$, where $t_a, t_b, t_c \in \mathbb{R}$ and $t_a t_b t_c \neq 0$ (Figure 4). As $\begin{vmatrix} 0 & 1 & t_a \\ t_b & 0 & 1 \\ 1 & t_c & 0 \end{vmatrix} = 1 + t_a t_b t_c$, then T_a, T_b, and T_c are collinear if and only if $t_a t_b t_c = -1$. Now, the equation of the line g given by the points T_a, T_b, and T_c is, ex., $\begin{vmatrix} x & y & z \\ 0 & 1 & t_a \\ t_b & 0 & 1 \end{vmatrix} = x + t_a t_b y - t_b z = 0$. Because the point $T(t+1 : t_c t : t_a t_b)$, $t \in \mathbb{R}$ is incident with g, then the parametric system of equations of the line g with parameter t can be considered as $x(t) = t+1$, $y(t) = t_c t$, $z(t) = t_a t_b$, and the coordinates of the Exeter transformation of the point T give the parametric system of equations of the Exeter transformation of the line g. Thus, using for T the Equation (2) of the Exeter transformation, we have

$$T_e = \Big(a^2(-\mathfrak{p} + \mathfrak{q} + \mathfrak{r}) : b^2(\mathfrak{p} - \mathfrak{q} + \mathfrak{r}) : c^2(\mathfrak{p} + \mathfrak{q} - \mathfrak{r})\Big), \tag{3}$$

where $\mathfrak{p} = a^4 t_a^2 t_b^2 t_c^2 t^2$, $\mathfrak{q} = b^4 t_a^2 t_c^2 (t+1)^2$, and $\mathfrak{r} = c^4 t_c^2 t^2 (t+1)^2$. As the degree of the polynomial $\mathfrak{r}$ in variable t is four ($\mathfrak{p}$ and $\mathfrak{q}$ are second degree polynomials), then all coordinates of T_e are polynomials in t having degree four. Thus, $g_e = ExTr(g)$ with points T_e, where $t \in \mathbb{R}$ is a fourth-order curve. Moreover, according to Theorem 2 the images of T_a, T_b and T_c are A_t, B_t, and C_t, respectively (Figure 4). □

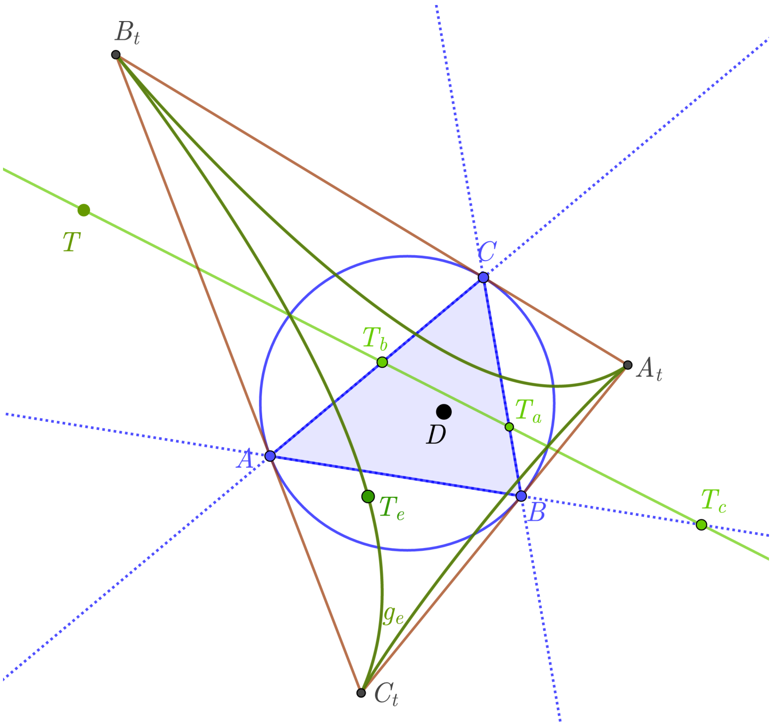

Figure 4. Exeter transformation of the line g.

2.1. Invariant Elements

Let D be the symmedian point of $\triangle ABC$ (X_6 in [1]). (The point of concurrence of lines AA_t, BB_t and CC_t. Thus, $\triangle ABC$ is perspective with $\triangle A_t B_t C_t$ at the point D. Furthermore, if $\triangle ABC$ is acute, then D is the Gergonne point of $\triangle A_tB_tC_t$, the point X_7 in [1].) It is well known that

$$D(a^2 : b^2 : c^2).$$

Theorem 4. *D is a fixed point of the Exeter transformation with respect to triangle ABC; moreover, $ExTr(D) = ExTr(A_t) = ExTr(B_t) = ExTr(C_t) = D$.*

Proof. It follows directly from Corollary 2. □

Theorem 5. *The lines AD, BD, and CD are invariant lines with respect to the Exeter transformations. Their images are parts of $\mathcal{R}$. Moreover, if $\triangle ABC$ is acute, then the Exeter transformations of lines AD, BD, and CD are the segments $AA_t \subseteq \mathcal{R}$, $BB_t \subseteq \mathcal{R}$, and $CC_t \subseteq \mathcal{R}$, respectively.*

Proof. For example, let P be on the line AD. As lines AD and AA_t are the same, then A' and P_e are on that line. Analytically, the equation of line AD is $c^2y - b^2z = 0$ and the coordinates of its arbitrary point P is $(t : b^2 : c^2)$, $t \in \mathbb{R}$ and its image with respect to (2) is $(a^2 - 2t^2/a^2 : b^2 : c^2)$, which is also on line AD.

The proof is similar for the case of the other lines. □

Theorem 6. *The circumcircle of triangle ABC is fixed (all of its points are fixed, except A, B, and C) over the Exeter transformation, so $ExTr(\mathcal{C}) = \mathcal{C}$.*

Proof. The barycentric equation of $\mathcal{C}$ is $a^2yz + b^2zx + c^2xy = 0$. If $P(u : v : w)$ is a point of $\mathcal{C}$, then from $a^2vw + b^2wu + c^2uv = 0$ we have

$$P(u : \frac{-b^2uw}{a^2w + c^2u} : w) = P(a^2uw + c^2u^2 : -b^2uw : a^2w^2 + c^2uw).$$

Using (2) we obtain $ExTr(P) = P$. Recall $P \neq B$, thus $a^2w + c^2u \neq 0$. □

The points A_t, B_t, C_t, and D determine a pencil of conic $\mathcal{Q}$. Let $\mathcal{Q}(P)$ denote the element of the pencil $\mathcal{Q}$ on which the point P lies. If $\triangle ABC$ is acute, then D is inside of the triangle $A_tB_tC_t$, and the conics of the pencil $\mathcal{Q}$ are hyperbolas.

Theorem 7. *The point and its image under the Exeter transformation with respect to triangle ABC lie on the same conic of the pencil $\mathcal{Q}$, so $\mathcal{Q}(P) = \mathcal{Q}(ExTr(P))$, and the equation of $\mathcal{Q}(P)$ is*

$$\boldsymbol{\alpha} x^2 + \boldsymbol{\beta} y^2 + \gamma z^2 = 0, \tag{4}$$

where

$$\boldsymbol{\alpha} = b^4w^2 - c^4v^2, \quad \boldsymbol{\beta} = -a^4w^2 + c^4u^2, \quad \gamma = a^4v^2 - b^4u^2 \tag{5}$$

and $(u : v : w)$ are the barycentric coordinates of P.

Proof. The equation of a conic is

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \alpha & \nu & \mu \\ \nu & \beta & \lambda \\ \mu & \lambda & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0. \tag{6}$$

Let $\gamma = 1$, without loss of generality. Substituting points D, A_t, B_t, C_t, and P into (6) we have a system of five linear equations and after a homogenization the solution gives the Equation (4) of $\mathcal{Q}(P)$. Moreover, the coordinates of $ExTr(P)$ satisfy the Equation (4), so the image of P is on $\mathcal{Q}(P)$. □

Remark 3. *If P is not on any sideline of $\triangle A_tB_tC_t$, so $\boldsymbol{\alpha\beta\gamma} \neq 0$, then the conic $\mathcal{Q}(P)$ is non-degenerate.*

Corollary 6. *All conics (or degenerate conics-crossing lines at points A, B, or C) lying through the points A_t, B_t, C_t and D are invariant. Furthermore, their fixed points are D and the intersection points with the circumcircle of $\triangle ABC$.*

Corollary 7. *The images of lines A_tB_t, B_tC_t, and C_tA_t are parts of the lines CC_t, AA_t, and BB_t, respectively. Moreover, in the case of an acute $\triangle ABC$ they are the segments CC_t, AA_t, and BB_t, respectively.*

Theorem 8. *The image of the pencil of a line through A is a pencil of a line through A_t and the corresponding lines intersect each other at the points of $\mathcal{C}$.*

Proof. It is a simple corollary of the definition of the Exeter transformation. □

Let the point sequence P_i ($i \geq 0$) be the ith image of P. Thus, $P_i = (ExTr)^i(P)$ and $P_0 = P$.

Corollary 8. *All elements of the point sequence P_i ($i \geq 0$) are on the same conic $\mathcal{Q}(P)$ (see Figure 5).*

Proof. Every conic is clearly defined by five points. The conic $\mathcal{Q}(P)$ is given by A_t, B_t, C_t, D, and $P = P_0$. From Theorem 7, we have that the point P_1, the image of the point P_0 under the Exeter transformation, lies on the conic $\mathcal{Q}(P)$. Thus, A_t, B_t, C_t, D, and P_1 also define $\mathcal{Q}(P)$. Recursively—using Theorem 7—we can prove that the point P_2 (and so $P_3, P_4, \ldots$) lies on the conic $\mathcal{Q}(P)$ as well. □

From Theorem 7 and Corollaries 2 and 8 we gain

Corollary 9. *The points $P^0 = P$, P^1, P^2 and P^3 are on the same conic $\mathcal{Q}(P)$.*

Proof. If $P = (u : v : w)$ and $P^i = (\pm u : \pm v : \pm w)$, $i = 0, 1, 2, 3$, then not only P satisfies the Equation (4) of the conic $\mathcal{Q}(P)$, but also each P^i does. □

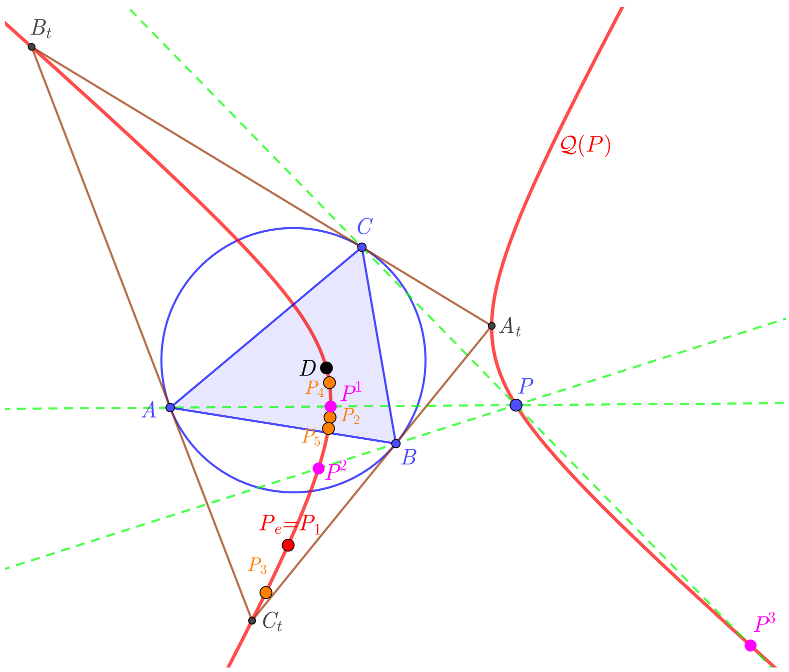

Figure 5. Points on conic $\mathcal{Q}(P)$.

Corollary 10. $P^1 = AP \cap \mathcal{Q}(P)$, $P^2 = BP \cap \mathcal{Q}(P)$, $P^3 = CP \cap \mathcal{Q}(P)$ *(See Figure 5).*

Proof. From Corollary 9, we know that, for example, the points $P^0 = P$ and P^1 are on the same conic $\mathcal{Q}(P)$. The point $P^1(-u : v : w)$ lies on the line AP with the equation $wy - vz = 0$. Thus, they are concurrent. As a line has maximum two intersection points with a non-degenerate conic, the other intersection point of the line AP with $\mathcal{Q}(P)$ must be P^1. □

Corollary 11. *The vertices of any complete quadrangle with diagonal points A, B, and C are on the same conic $\mathcal{Q}$.*

Proof. According to Corollary 10 and Corollary 5 we know that P^i, $i = 0, 1, 2, 3$ lie on the same conic and (clearly) define a complete quadrangle with diagonal points A, B and C. Moreover, it is also well known that a complete quadrangle is clearly defined by three diagonal points and a vertex, i.e., A, B, C, and any point P, where P does not lie on the sidelines of the triangle ABC. The corollary summarizes these statements. □

2.2. Tangent Lines

Let $\mathcal{Q}(P)$ be a non-degenerate conic of the pencil $\mathcal{Q}$. Let us denote by t_X the tangent line to $\mathcal{Q}(P)$ at a point $X \in \mathcal{Q}(P)$.

Theorem 9. *If the points P^i, $i = 0, 1, 2, 3$ are mapped onto the same point by the Exeter transformation with respect to triangle ABC, then the intersection points of the tangent lines t_{p0}, t_{p1}, t_{p2}, and t_{p3} are on the lines of the sides of triangle ABC.*

Proof. Let us consider an arbitrary line with barycentric equation $px + qy + rz = 0$. If it is passing through the point $P = P^0$, then $-pu = qv + rw$. Moreover, the equation of the pencil of lines at P^0 is

$$(-qv - rw)x + uqy + urz = 0, \tag{7}$$

where q and r are the barycentric coordinates of the lines from the pencil. To derive the tangent line among them, we consider the system of Equations (4) and (7). If its discriminant is zero and we put $x = 1$, then using (5) we have $q = (v\boldsymbol{\beta}/w\boldsymbol{\gamma})r$. Finally, the equation of t_{P_0} is

$$-\left(v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma}\right)x + uv\boldsymbol{\beta}\,y + uw\boldsymbol{\gamma}\,z = 0.$$

Similarly, the equations of the pencil of lines at P^1, P^2 and P^3, respectively, are $(qv + rw)x + uqy + urz = 0$, $(qv - rw)x + uqy + urz = 0$ and $(-qv + rw)x + uqy + urz = 0$. Moreover, the equations of lines t_{p1}, t_{p2}, and t_{p3}, respectively, are $\left(v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma}\right)x + uv\boldsymbol{\beta}\,y + uw\boldsymbol{\gamma}\,z = 0$, $\left(v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma}\right)x + uv\boldsymbol{\beta}\,y - uw\boldsymbol{\gamma}\,z = 0$ and $\left(v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma}\right)x - uv\boldsymbol{\beta}\,y + uw\boldsymbol{\gamma}\,z = 0$.

Let the point R^{01} be the intersection of t_{p0} and t_{p1}. From the equations of lines we have that $R^{01} = (0 : w\boldsymbol{\gamma} : -v\boldsymbol{\beta})$, which is on the line BC (Figure 6).

The proof is similar in the case of the other intersection points. Moreover, the intersection points are $R^{02} = \left(uw\boldsymbol{\gamma} : 0 : v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma}\right)$, $R^{03} = \left(uv\boldsymbol{\beta} : v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma} : 0\right)$, $R^{12} = \left(-uv\boldsymbol{\beta} : v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma} : 0\right)$, $R^{13} = \left(-uw\boldsymbol{\gamma} : 0 : v^2\boldsymbol{\beta} + w^2\boldsymbol{\gamma}\right)$, and $R^{23} = (0 : w\boldsymbol{\gamma} : v\boldsymbol{\beta})$. □

Remark 4. *The points R^{ij} and A, B, and C define a complete quadrilateral with diagonal points and projective harmonic conjugate point pairs with respect to the diagonal points. For example, in Figure 6, the points R^{12}, R^{03}, R^{02}, and R^{13} form a complete quadrilateral with diagonal points A, R^{23}, R^{01}; B which is the projective harmonic conjugate of A with respect to R^{03} and R^{12}; and C is the projective harmonic conjugate of A with respect to R^{02} and R^{13}. Their cross-ratio is -1.*

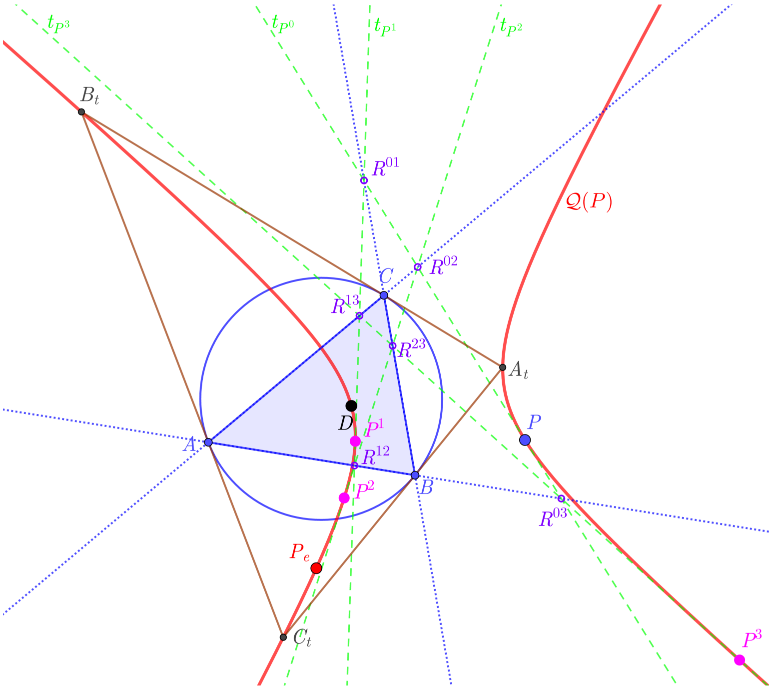

Figure 6. Tangent lines at points P^i.

Corollary 12. *The intersection of any two lines t_{A_t}, t_{B_t}, t_{C_t}, and t_D are on one of the sidelines of triangle ABC.*

Proof. Considering Theorem 4, we obtain the statement from Theorem 9 (see Figure 7). Moreover, the equations of $t_{A_t}, t_{B_t}, t_{C_t}$, and t_D, respectively, are $(b^4\boldsymbol{\beta} + c^4\gamma)x + a^2b^2\boldsymbol{\beta}y + a^2c^2\gamma z = 0$, $(b^4\boldsymbol{\beta} + c^4\gamma)x + a^2b^2\boldsymbol{\beta}y - a^2c^2\gamma z = 0$, $(b^4\boldsymbol{\beta} + c^4\gamma)x - a^2b^2\boldsymbol{\beta}y + a^2c^2\gamma z = 0$, and $-(b^4\boldsymbol{\beta} + c^4\gamma)x + a^2b^2\boldsymbol{\beta}y + a^2c^2\gamma z = 0$. Their intersection points are $T_{AB} = \left(-a^2b^2\boldsymbol{\beta} : b^4\boldsymbol{\beta} + c^4\gamma : 0\right)$, $T_{AC} = \left(-a^2c^2\gamma : 0 : b^4\boldsymbol{\beta} + c^4\gamma\right)$, $T_{AD} = \left(0 : c^2\gamma : -b^2\boldsymbol{\beta}\right)$, $T_{BC} = \left(0 : c^2\gamma : b^2\boldsymbol{\beta}\right)$, $T_{AD} = \left(a^2c^2\gamma : 0 : b^4\boldsymbol{\beta} + c^4\gamma\right)$, $T_{CD} = \left(a^2b^2\boldsymbol{\beta} : b^4\boldsymbol{\beta} + c^4\gamma : 0\right)$. □

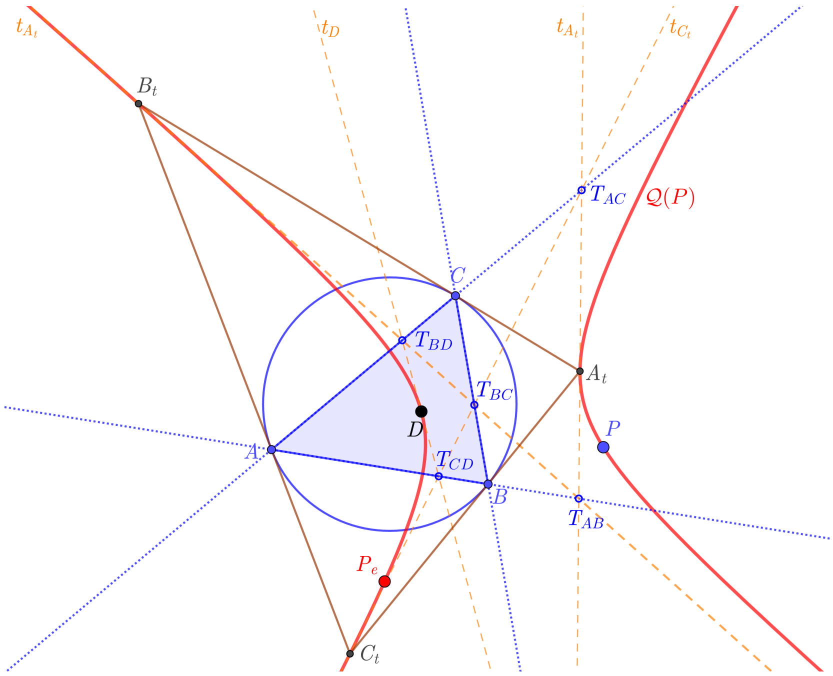

Figure 7. Tangent lines at points A_t, B_t, C_t, and D.

Let M_{AB1} and M_{AB2} be the intersection points of line AB and the non-degenerate $\mathcal{Q}(P)$. Let us define similarly the points M_{AC1}, M_{AC2}, M_{BC1}, and M_{BC2}. Their coordinates from Equation (4) are $M_{AB1}(\sqrt{-\frac{\beta}{\alpha}} : 1 : 0)$, $M_{AB2}(\sqrt{-\frac{\beta}{\alpha}} : -1 : 0)$, $M_{AC1}(1 : 0 : \sqrt{-\frac{\alpha}{\gamma}})$, $M_{AC2}(-1 : 0 : \sqrt{-\frac{\alpha}{\gamma}})$, $M_{BC1}(0 : \sqrt{-\frac{\gamma}{\beta}} : 1)$, and $M_{BC2}(0 : \sqrt{-\frac{\gamma}{\beta}} : -1)$. (Recall $\alpha\beta\gamma \neq 0$.) Naturally, $\mathcal{Q}(P)$ does not intersect the lines of the sides of $\triangle ABC$ at the same time, for example, α could not be positive and negative at the same time. Thus, at most four points exist at the same time among the above points. See Figure 8, when $\mathcal{Q}(P)$ intersects the lines AC and AB, so $\frac{\alpha}{\gamma} < 0$, $\frac{\gamma}{\beta} < 0$ and then $\frac{\beta}{\alpha} < 0$.

Theorem 10. *The tangent lines to $\mathcal{Q}(P)$ at points M_{AB1} and M_{AB2} are passing through the point C. Similarly, the tangent lines to $\mathcal{Q}(P)$ at points M_{AC1}, M_{AC2} or M_{BC1}, M_{BC2} are passing through the points B or A, respectively.*

Proof. The equations of the tangent lines at points M_{AB1} and M_{AB2} are $-x + \sqrt{-\frac{\beta}{\alpha}}y = 0$ and $x + \sqrt{-\frac{\beta}{\alpha}}y = 0$, respectively. Point C coincides with them. □

Let M_1, M_2, M_3, and M_4 be the intersection points different from A, B, and C of the tangent lines at M_{AB1}, M_{AB2}, M_{AC1}, M_{AC2}, M_{BC1}, and M_{BC2}.

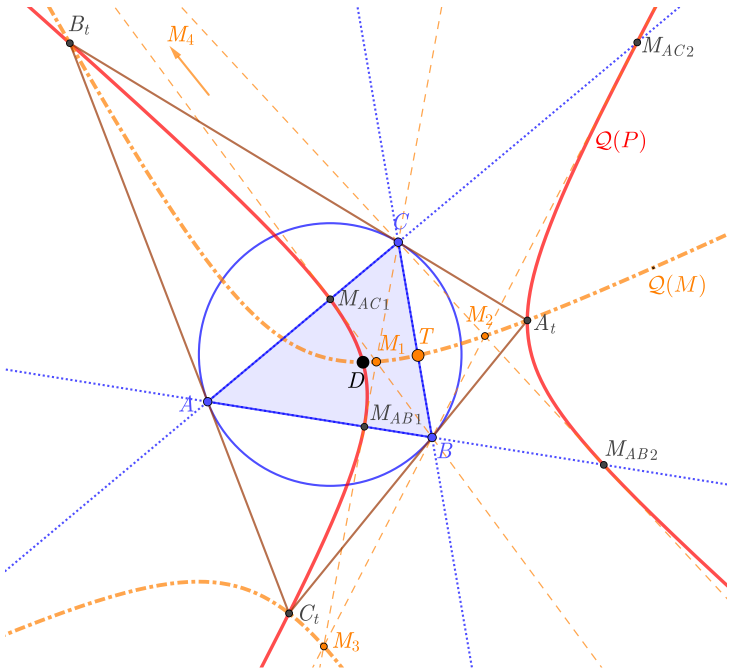

Figure 8. Tangent lines at intersection points.

Theorem 11. *The points M_1, M_2, M_3 and M_4 are on the same conic $\mathcal{Q}(M)$ with one of the equations*

$$\alpha\left(b^4\beta - c^4\gamma\right)x^2 - \beta\left(a^4\alpha + c^4\gamma\right)y^2 + \gamma\left(a^4\alpha + b^4\beta\right)z^2 = 0, \tag{8}$$

$$-\alpha\left(b^4\beta + c^4\gamma\right)x^2 + \beta\left(a^4\alpha + c^4\gamma\right)y^2 + z^2\gamma\left(a^4\alpha - b^4\beta\right) = 0, \tag{9}$$

$$\alpha\left(b^4\beta + c^4\gamma\right)x^2 + \beta\left(-a^4\alpha + c^4\gamma\right)y^2 - z^2\gamma\left(a^4\alpha + b^4\beta\right) = 0, \tag{10}$$

when $\mathcal{Q}(M)$ does not intersect lines BC, AB, or AC, respectively.

Proof. First, we consider the case of Figure 8. The points M_1, M_2, M_3, and M_4 define a complete quadrangle with diagonal points A, B, and C and according to Corollary 11 they are on the same conic of $\mathcal{Q}$. The barycentric coordinates of M_i are $\left(\pm\sqrt{-\frac{\gamma}{\alpha}} : \pm\sqrt{\frac{\gamma}{\beta}} : 1\right)$. The Equation (8) is determined similarly to the equation of $\mathcal{Q}(P)$ in Theorem 7.

Second, when $\mathcal{Q}(M)$ does not intersect the lines AB or AC, respectively, the coordinates of M_i are $\left(\pm\sqrt{-\frac{\gamma}{\alpha}} : \pm\sqrt{-\frac{\gamma}{\beta}} : 1\right)$ or $\left(\pm\sqrt{\frac{\gamma}{\alpha}} : \pm\sqrt{-\frac{\gamma}{\beta}} : 1\right)$. They yield the Equations (9) and (10). □

Theorem 12. *The Exeter transformations of M_1, M_2, M_3, and M_4 are the same point T lying on one of the lines of $\triangle ABC$. Moreover, T coincides with T_{AB}, T_{AC}, or T_{BC}, when $\mathcal{Q}(P)$ does not intersect the lines BC, AB, or AC, respectively.*

Proof. We use Corollaries 2 and 5. Analytically, for example, in case of Figure 8, $ExTr(M_1) = ExTr(M_2) = ExTr(M_3) = ExTr(M_4) = (0 : c^2\gamma : b^2\beta) = T_{BC} = T$. □

Let $\mathcal{N}$ be the nine-point conic determined by points A_t, B_t, C_t, and D [10]. Conic $\mathcal{N}$ is passing through the points A, B, C, and the midpoints of all segments of points A_t, B_t, C_t, and D. Moreover, it also well known that the center of a conic passing through points A_t, B_t, C_t, and D is on the nine-point conic $\mathcal{N}$ [10]. This proves the following theorem.

Theorem 13. *The center of the conic $\mathcal{Q}(P)$ is lying on the conic $\mathcal{N}$ (Figure 9).*

With a short calculation, we have $a^4yz + b^4xz + c^4xy = 0$ for the equation of $\mathcal{N}$ and the coordinates of the center of $\mathcal{Q}(P)$ are

$$(\beta\gamma : \alpha\gamma : \alpha\beta).$$

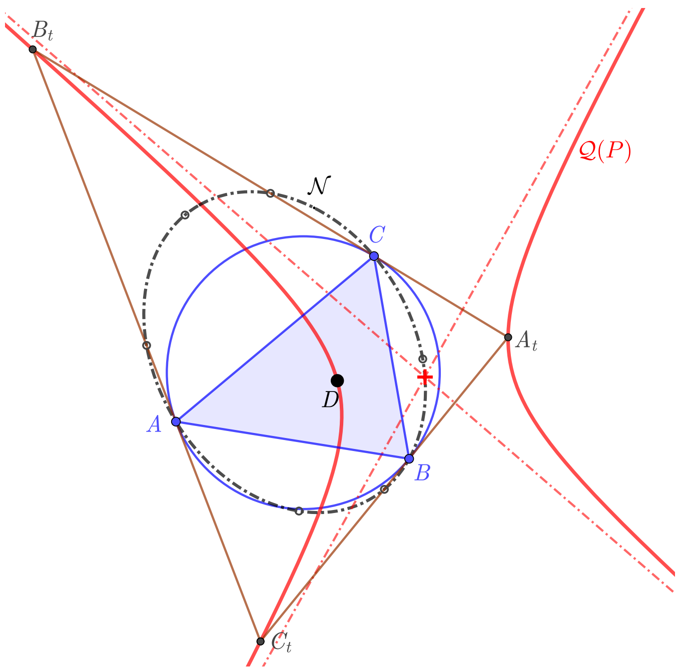

Figure 9. Nine-point conic.

3. Conclusions

With the help of the barycentric coordinates, we showed that the extension of the well-known process of the Exeter point from the centroid of a given triangle ABC provides a so-called Exeter transformation for the whole plane. Each point P, its image P_e, the symmedian, and three exsymmedian points of the triangle are on the same conic $\mathcal{Q}(P)$. Moreover, three other points (P^1, P^2, and P^3) of this conic $\mathcal{Q}(P)$ have the common image point P_e as well. The Exeter transformation is nonlinear, as the image of a general line is a fourth-order curve passing through the exsymmedian points of the triangle. We presented the images of some special lines, points, and we determined the invariant points and lines of the Exeter transformation. We particularly examined the intersection points of the tangent lines of the conic $\mathcal{Q}(P)$ at some special points. We think that this nonlinear transformation will reveal several interesting and useful properties during future investigations.

Author Contributions: Conceptualization, P.C. and L.N.; Funding acquisition, P.C.; Investigation, P.C. and L.N.; Methodology, P.C. and L.N.; Software, P.C. and L.N.; Visualization, P.C. and L.N.; Writing—original draft, P.C. and L.N.; Writing—review and editing, P.C. and L.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by VEGA grant (Vedecká Grantová Agentúra MŠVVaŠ SR) number 1/0386/21.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kimberling, C. Encyclopedia of Triangle Centers. Available online: http://faculty.evansville.edu/ck6/encyclopedia/ETC.html (accessed on 8 December 2020).
2. Minevich, I.; Morton, P. Synthetic foundations of cevian geometry, IV: The TCC-Perspector Theorem. *Int. J. Geom.* **2017**, *6*, 61–85.
3. Grozdev, S.; Okumura, H.; Dekov, D. Incentral triangle. *Comput. Discov. Math.* **2017**, *2*, 35–45.
4. Grozdev, S.; Okumura, H.; Dekov, D. Notable circles. *Comput. Discov. Math.* **2017**, *2*, 117–134.
5. Kimberling, C. Exeter Point. Available online: https://faculty.evansville.edu/ck6/tcenters/recent/exeter.html (accessed on 8 December 2020).
6. Scott, J.A. The Exeter point revisited. *Math. Gaz.* **2012**, *96*, 160–161. [CrossRef]
7. Capitan, F.J.G. Barycentric Coordinates. *Int. J. Comput. Discov. Math.* **2015**, *0*, 32–48. Available online: https://www.journal-1.eu/2015/01/Francisco-Javier-Barycentric-Coordinates-pp.32-48.pdf (accessed on 8 December 2020).
8. Volenec, V. Metrical relations in barycentric coordinates. *Math. Commun.* **2003**, *8*, 55–68.
9. Volenec, V. Circles in barycentric coordinates. *Math. Commun.* **2004**, *9*, 79–89.
10. Pierce, D. Thales and the nine-point conic. *Morgan Gaz.* **2016**, *8*, 27–78.