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Sequences related to square and cube zig-zag shapes

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The full version of this work partly can be found in [8] and partly will be published elsewhere.

Abstract

Considering a so-called square k-zig-zag shape as a part of the regular square grid as a k-zig-zag digraph, we give values to its vertices according to the number of the shortest paths from a base vertex. It provides several integer sequences, whose higher-order homogeneous recurrences are determined by the help of a special matrix recurrence. We also define a special zig-zag shape based on the spatial cube grid, and we give recurrence relation for one of their digraph walk.

1 Introduction

The present extended abstract summarizes the studies of diagonal and zig-zag paths on a particular $k+1$ wide, infinite part of the usual square lattice, and along these paths we determine linear recurrence sequences that are mostly defined in the *On-Line Encyclopedia of Integer Sequences* (OEIS, [9]) without combinatorial interpretations. In this manner our investigation, among others, gives them geometrical and combinatorial background. The consideration of zig-zag shapes is not an isolated challenge. For example, Baryshnikov and Romik [2] examined the so-called Young diagrams, which are similar to our construction, and defined a kind of 'zig-zag' numbers by the help of the alternating permutations. Stanley [10] published a survey in which he dealt with the 'zig-zag' shapes and the alternating permutations. Recently, Ahmad et al. [1] studied some graph-theoretic properties of special zig-zag polyomino chains.

The authors proved in [3, 6] that all the integer linear homogeneous recurrence sequences $\{f_i\}_{i\geq 0}$ defined by

$$f_i = \alpha f_{i-1} \pm f_{i-2}, \qquad (i \geq 2),$$

where $\alpha \in \mathbb{N}$, $\alpha \geq 2$, and $f_0 < f_1$ are positive integers with $\gcd(f_0, f_1) = 1$, appear along corresponding zig-zag paths in the hyperbolic Pascal triangle $\{4, 5\}$. Moreover, in a special case the Fibonacci sequence appears, as well. This interesting result also inspired us to examine zig-zag paths on certain parts of the Euclidean square mosaic.

Consider the Euclidean square lattice and take k consecutive pieces of squares. This is the 0th layer of the k–zig-zag shape. The upper corners are the 1st, 2nd, ..., kth and $(k+1)$st vertices according to Figure 1. Extend this by an extra 0th vertex, which is the base vertex. We color it by yellow in the figures, and we join it to the 1st vertex by an extra edge. We denote the vertices of the 0th line by small boxes in Figure 1. Now move the 0the layer to reach the right-down position in the square lattice to obtain the 1st layer, and repeat this procedure with the latest layer infinitely many times. Thus, we define the square k–zig-zag shape or graph, where $k \geq 1$ is the size of the array. Finally, we label the vertices such that a label gives the number of different shortest paths from the base vertex. Figure 2 illustrates the first few layers of the square 4–zig-zag digraph, the vertices are denoted by shaded boxes with their label values and the directed edges are the black arrows. (Certain black arrows are re-colored by red for some reason. There are also particular blue arrows in the Figure; their role will be discussed

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later.) Let $a_{i,j}$ denote the label of the vertex located in ith row and jth position ($0 \le j \le k+1$, $0 \le i$). Clearly, the fundamental rule of the construction is given by

$$a_{i,j} = \begin{cases} 1, & \text{if } i = 0; \\ a_{i-1,1}, & \text{if } j = 0, 1 \le i; \\ a_{i,j-1} + a_{i-1,j+1}, & \text{if } 1 \le j \le k, 1 \le i; \\ a_{i,k}, & \text{if } j = k+1, 1 \le i. \end{cases} \tag{1}$$

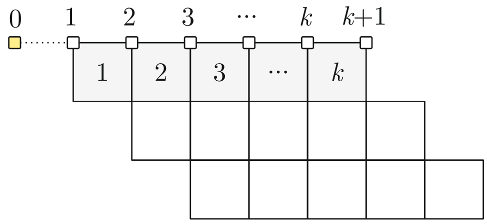

Figure 1: Zig-zag shape

For fixed $k \ge 1$ and given $0 \le j \le k+1$, let $A_j^{(k)}$ be the sequence defined by $A_j^{(k)} = (a_{i,j})_{i=0}^{\infty}$. The sequence $A_j^{(k)}$ is the jth right-down diagonal sequence of the square k–zig-zag shape. In Figure 2, the blue arrows represent the sequence $A_1^{(4)}$. We found $A_0^{(k)} = (1, A_1^{(k)})$ and $A_k^{(k)} = A_{k+1}^{(k)}$.

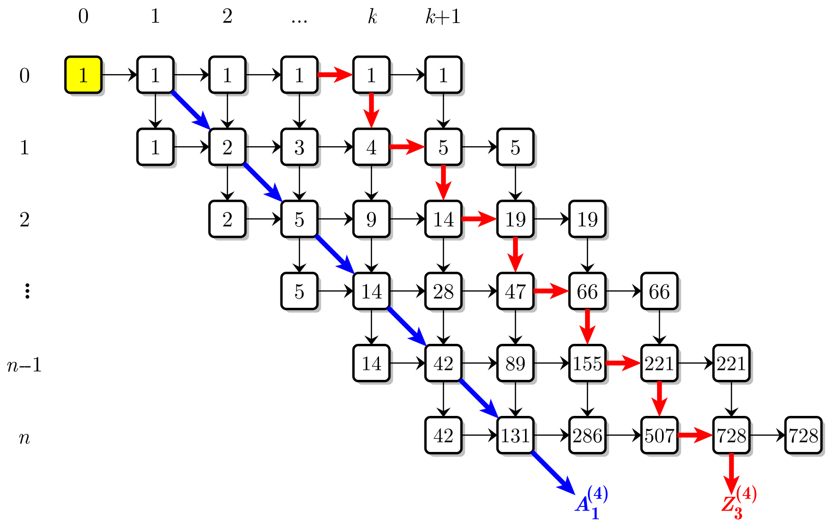

Figure 2: Square 4–zig-zag digraph ($k = 4$)

Let $Z_j^{(k)}$, $j \in \{0, 1, \ldots, k\}$ be the jth zig-zag sequence of the square k–zig-zag shape, where $Z_j^{(k)}$ is the merged sequence of $A_j^{(k)}$ and $A_{j+1}^{(k)}$. (In Figure 2, the red arrows represent the zig-zag sequence $Z_3^{(4)}$.) More precisely, $Z_j^{(k)} = (z_{i,j})_{i=0}^{\infty}$, where

$$z_{i,j} = \begin{cases} a_{\ell,j}, & \text{if } i = 2\ell; \\ a_{\ell,j+1}, & \text{if } i = 2\ell+1. \end{cases} \tag{2}$$

Since $Z_0^{(k)}$ and $Z_k^{(k)}$ are the 'double' of $A_0^{(k)}$ and $A_k^{(k)}$, respectively, usually we examine sequences for $j \in \{1, 2, \ldots, k-1\}$.

Now we record the two main theorems of this paper. The second one is a simple corollary of the first one.

Theorem 1 (Main theorem)**.** *Given $k \geq 1$. Then all the right-down diagonal sequences $A_j^{(k)}$ for $j \in \{0, 1, \dots, k, k+1\}$ have the same $(\left\lfloor \frac{k}{2} \right\rfloor + 1)$-th order homogeneous linear recurrence relation*

$$a_{n,j} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^i \binom{k+1-i}{i+1} a_{n-1-i,j}, \qquad n \geq \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

Theorem 2. *Fixing $k \geq 1$, the zig-zag sequences $Z_j^{(k)}$ for $j \in \{0, 1, \dots, k\}$ satisfy a $(2\left\lfloor \frac{k}{2} \right\rfloor + 2)$-th order homogeneous linear recurrence relation given by*

$$z_{n,j} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^i \binom{k+1-i}{i+1} z_{n-1-2i,j}, \qquad n \geq 2\left\lfloor \frac{k}{2} \right\rfloor + 2.$$

2 Recurrence relations of the square zig-zag shapes

We find that any item $a_{n,j}$, $(n \geq 1)$ is the sum of the certain items of $(n-1)$st row. More precisely, if $0 < j < k+1$, then

$$a_{n,j} = a_{n-1,j+1} + a_{n,j-1} = a_{n-1,j+1} + a_{n-1,j} + a_{n,j-2} = \cdots = \sum_{\ell=1}^{j+1} a_{n-1,\ell}. \tag{3}$$

Consider (3) for all $j \in \{1, 2, \dots, k+1\}$ we obtain the system

$$\mathbf{v}_n = \mathbf{M} \cdot \mathbf{v}_{n-1}, \quad n \geq 1, \tag{4}$$

where

$$\mathbf{v}_n = \begin{pmatrix} a_{n,1} \\ a_{n,2} \\ a_{n,3} \\ \vdots \\ a_{n,k} \\ a_{n,k+1} \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{M}^{(k+1)\times(k+1)} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

We know from [5] and [7] that the characteristic polynomial of any recurrence sequence (r) defined by the linear combination of the recurrence sequences $(a_j) = (a_{i,j})_{i \geq 0}$ of system (4), moreover the characteristic polynomial of the coefficient matrix $\mathbf{M}$ of system (4), coincide. Consequently, we have to determine the characteristic polynomial $p_k(x)$ of $\mathbf{M}$, and then $p_k(x)$ yields the common recurrence relation of the sequences $\{a_j\}$ and their linear combinations.

Since

$$p_k(x) = |x\mathbf{I} - \mathbf{M}|,$$

where $\mathbf{I}$ is the appropriate unit matrix, we obtain

$$\begin{aligned} p_0(x) &= x - 1, \\ p_1(x) &= \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = x^2 - 2x, \end{aligned}$$

and for $p_k(x)$ some calculation (for details see [8]) we have the binary recurrence relation

$$p_k(x) = x \cdot p_{k-1}(x) - x \cdot p_{k-2}(x), \quad k \geq 2. \tag{5}$$

Because each recurrence coefficient in (5) is one of $\pm x$, the factorization of $p_k(x)$ contains a factor x^m for some positive integer m. The next theorem provides, among others, the precise exponent m in the factorization of $p_k(x)$.

Theorem 3. *The characteristic polynomials $p_k(x)$ can be given by*

$$p_k(x) = x^{\lceil \frac{k}{2} \rceil} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor + 1} (-1)^i \binom{k+2-i}{i} x^{\lfloor \frac{k}{2} \rfloor + 1 - i}, \qquad k \geq 0.$$

By the help of Theorem 3 we are ready to give the recurrences of the right-down diagonal sequences $A_j^{(k)}$ given Theorem 1.

2.1 Sum of rows, columns, and left-down diagonal sequences

Let $R^{(k)} = (r_n^{(k)})$ be the sum sequence of the values of the nth row of square k–zig-zag shape (see Figure 3). Considering the partial sum relation (3) we obtain

$$r_n^{(k)} = \sum_{j=0}^{k+1} a_{n,j} = a_{n,0} + a_{n+1,k}.$$

So, the recurrence sequence $r_n^{(k)}$ is the linear combination of sequences $A_n^{(k)}$, therefore they have the same characteristic polynomial and the same recurrence relation.

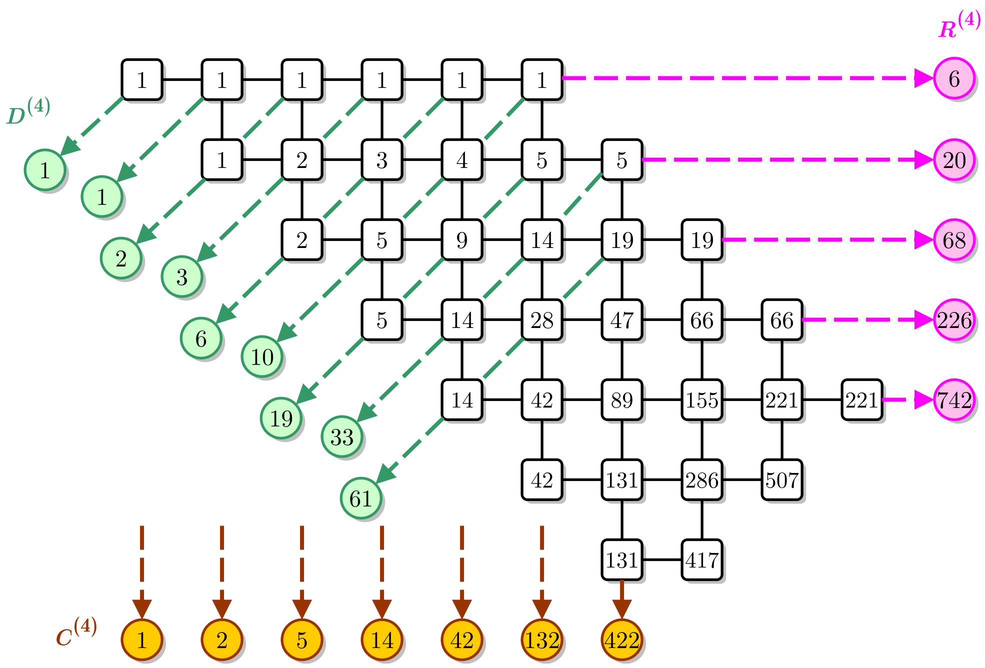

Figure 3: Sum of rows, columns, and left-down diagonal sequences

Let $C^{(k)} = (c_n^{(k)})$ be the sum sequence of columns. As $a_{n+1,0} = a_{n,1} = a_{n,0} + a_{n-1,2} = a_{n,0} + a_{n-1,1} + a_{n-2,3} = \cdots = \sum_{j=0}^{\min\{n,k+1\}-1} a_{n-j,j}$, then

$$c_n^{(k)} = \sum_{j=0}^{\min\{n,k+1\}} a_{n-j,j} = \begin{cases} a_{n+1,0}, & \text{if } n \leq k; \\ a_{n+1,0} + a_{n-k-1,k+1}, & \text{if } n > k. \end{cases}$$

Let $D^{(k)} = (d_n^{(k)})$ be the left-down diagonal sequence, where

$$d_n = \begin{cases} \displaystyle\sum_{\ell=0}^{\ell \leq \frac{n}{2},\, 2\ell \leq k} a_{\frac{n}{2}-\ell, 2\ell}, & \text{if } n \text{ is even;} \\ \displaystyle\sum_{\ell=0}^{\ell \leq \frac{n}{2},\, 2\ell \leq k+1} a_{\frac{n}{2}-\ell, 2\ell+1}, & \text{if } n \text{ is odd.} \end{cases}$$

Since all the $A_j^{(k)}$ sequence satisfy the same recurrence relation, then $C^{(k)}$ and $D^{(k)}$ are so.

3 Spacial zig-zag cube graphs

Now, we define a chain of cubes as an infinite part of the cube grid in the 3-dimensional space. Given a cube as the first item of the chain. Chose one of its vertices as a base vertex of our construction. (This vertex is denoted by a_0 in the righ-hand side of Figure 4.) Let the second cube be the cube having a common face with the first and having the base vertex, then let the third be the cube having a common face with the second and a common edge with the first. Let the fourth one have a common face, edge and only one vertex with the third, second and first one, respectively, and so on. That way, generally, the nth cube has exactly one common face, edge, vertex with the $(n-1)$th, $(n-2)$th, $(n-3)$th cube of the chain, respectively. Now we associate the vertices with positive integer, which gives the numbers of the shortest ways to this vertex from the base vertex of the first cube. For the first eight cubes and the values of vertices of the chain, see the left-hand-side of Figure 4.

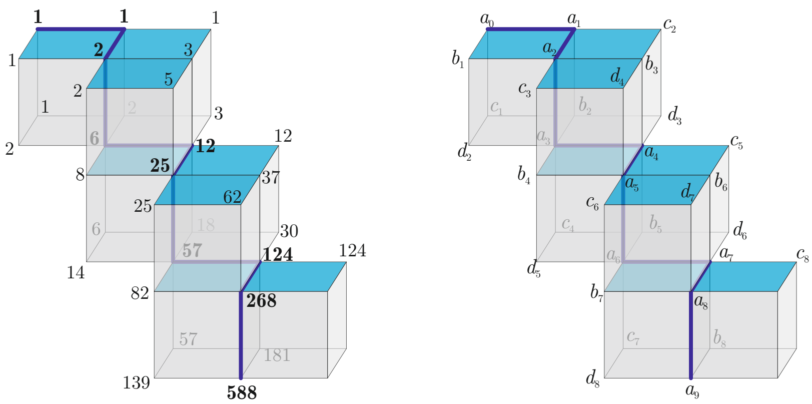

Figure 4: A cube zig-zag shape

According to the right-hand side of Figure 4 we can define zig-zag sequences a_i, b_i, c_i and d_i associated to the vertices of the cube chain. When we reconsider the vertices, edges and their sequences to a directed graph form, we gain Figure 5. Solving the system of recurrence equations, we obtain Theorem 4.

Theorem 4. *The sequence* $(a_i)_{i=0}^{\infty}$ *satisfies the fourth-order linear homogeneous recurrence relation*

$$a_{i+1} = a_i + a_{i-1} + 3a_{i-2} + a_{i-3}, \quad (i \geq 3)$$

with initial values $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$.

4 Conclusions

With a special directed zig-zag graph defined on a part of the square grid we generally determined the recurrences of some special directional sequences associated to the graph. We give new combinatorial

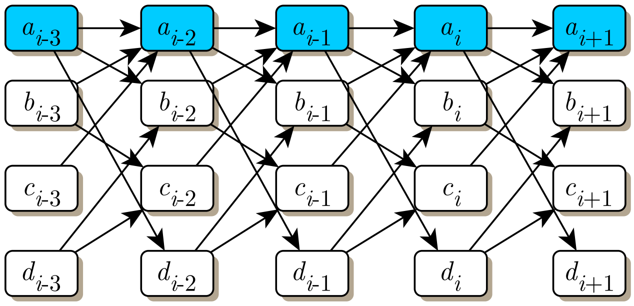

Figure 5: Digraph form of the system of recurrences

interpretations to more then forty sequences appearing in OEIS [9]. For example, if $k = 4$, then $A_1^{(4)} =$ A080937 $A_2^{(4)} =$ A094790, $A_3^{(4)} =$ A094789, $A_4^{(4)} =$ A005021, $Z_2^{(4)} =$ A006053, $(d_{2n}^{(4)}) =$ A052975, $(d_{2n+1}^{(4)}) =$ A060557, $D^{(4)} =$ A028495, and in case of cube zig-zag sequences: A214663, A232162, A232164, A232165.

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