Annales Mathematicae et Informaticae
43 (2014) pp. 113–121
http://ami.ektf.hu

Coincidences in numbers of graph vertices corresponding to regular planar hyperbolic mosaics

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Submitted September 02, 2014 — Accepted October 20, 2014

Abstract

The aim of this paper is to determine the elements which are in two pairs of sequences linked to the regular mosaics $\{4,5\}$ and $\{p,q\}$ on the hyperbolic plane. The problem leads to the solution of diophantine equations of certain types.

Keywords: regular planar hyperbolic mosaics, linear recurrences, diophantine equations.

MSC: 11B37, 51M10.

1. Introduction

Consider a regular mosaic on the hyperbolic plane. Such a mosaic is characterized by the Schläfli's symbol $\{p,q\}$. It is known that we can define belts of cells around a given vertex of the mosaic (see [4]). Let's say that belt $\mathcal{B}_0$ is the aforesaid fixed vertex itself denoted by B_0. The first belt $\mathcal{B}_1$ consists of the cells which connect to B_0. Assume now that the belts $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ are known ($i \geq 1$). Let belt $\mathcal{B}_{i+1}$ be created by the cells that have common point (not necessarily common vertex) with $\mathcal{B}_i$, but not with $\mathcal{B}_{i-1}$. Figure 1 shows the first three belts in the mosaic corresponding to $\{4,5\}$. One important question is to study the phenomenon of the growing of belts ([1], [2], [3]), even in higher dimensions, too.

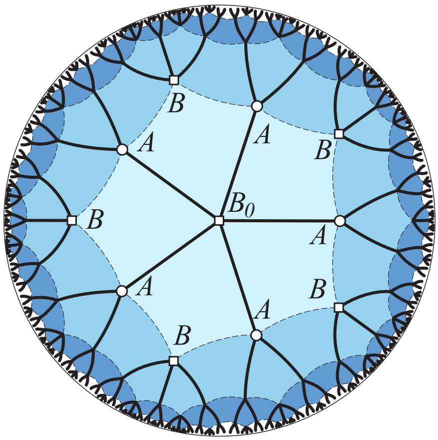

Figure 1: *Trees of the mosaic* $\{p, q\} = \{4, 5\}$

Take vertex B_0 as a main root of a will-be-graph (this is level 0). In general, let the outer boundary of belt $\mathcal{B}_i$ be called level i. Connect the vertices of level 1 to B_0 along the edges between the two levels of the lattice. By this way we have started to build trees. Then use always the maximum number of edges between level $(i-1)$ and level i. All vertices on level i are connected to only one vertex of the previous level, such that no unconnected leaves on level $(i-1)$ are remained. We never connect edges on the same level. The rest vertices on layer i will be roots of new trees. In this way, we obtain infinitely many trees, each of them contains infinitely many vertices. Let $\bar{A}$ denote the set of roots and $\bar{B}$ the set of other vertices. In Figure 1 and 2 the thick edges show the trees from level 0 to level 4. (We remark, that the dual problem is when we establish trees by connecting the centres of the cells of the mosaic.)

The case $q = 3$ provides no any tree since only one edge is not enough to connect the consecutive levels. If $p = 3$ the algorithm, apart from B_0, does not give roots. Therefore we may assume $p \geq 4$, $q \geq 4$, and since $(p-2)(q-2) = 4$ is the Euclidean lattice we also suppose $(p-2)(q-2) > 4$.

Let a_i and b_i denote the number of the vertices of $\bar{A}$ and $\bar{B}$ on level i, respectively. In this paper, we compare the terms a_i (and later b_i) of sequences belonging to different Schläfli's symbols $\{p, q\}$.

In the following, we recall some properties of the sequences a_i and b_i corresponding to hyperbolic planar lattice $\{p, q\}$ (see [4]). Simple geometric consideration shows $a_1 = q$, $b_1 = (p-3)q$, further the recursive system

$$a_n = (q-3)a_{n-1} + (q-2)b_{n-1}, \tag{1.1}$$

$$b_n = \big((q-3)(p-3)-1\big)a_{n-1} + \big((q-2)(p-3)-1\big)b_{n-1} \tag{1.2}$$

holds $(n \geq 2, p \geq 4, q \geq 4)$.

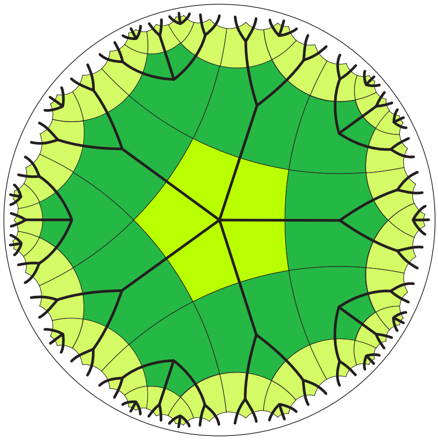

Figure 2: *Trees of the mosaic* $\{5, 4\}$*, dual of mosaic* $\{4, 5\}$

It is easy to separate the sequences $\{a_n\}$ and $\{b_n\}$, and it turns out that

$$a_n = \kappa a_{n-1} - a_{n-2} \qquad \text{and} \qquad b_n = \kappa b_{n-1} - b_{n-2}, \tag{1.3}$$

where $\kappa = (p-2)(q-2)-2$ $(\kappa \geq 4)$. Thus both sequences satisfy the same recurrence relation of order two, and they differ in their initials values. Indeed, to use (1.3) we need also the terms a_2 and b_2. Obviously, by (1.1) and (1.2), $a_2 = (\kappa+1)q$, $b_2 = (\kappa(p-3)-1)q$, and $(a_1, a_2) \neq (b_1, b_2)$. Later we also use the term $a_3 = (\kappa^2+\kappa-1)q$. Although a_0 and b_0 have no geometrical meaning, (1.3) provides the values $a_0 = -q$, $b_0 = q$, and this sometimes makes the calculations easier.

To achieve the investigations, we introduce the sufficient notations and recall some facts from the theory of linear recurrences. In general, let r and s denote arbitrary complex numbers. The sequence $\{G\}_{n=0}^{\infty}$ given by the initial values $G_0 \in \mathbb{C}$ and $G_1 \in \mathbb{C}$, and by the recursive relation

$$G_n = rG_{n-1} + sG_{n-2} \qquad (n \geq 2), \tag{1.4}$$

is called binary recurrence. For brevity, we often write $G(r, s, G_0, G_1)$ to indicate the parameters of the sequence $\{G\}$.

For any binary recurrence $G(r, s, G_0, G_1)$, the associate sequence of $\{G\}$ is the sequence $H(r, s, H_0, H_1)$ with

$$H_0 = 2G_1 - rG_0 \qquad \text{and} \qquad H_1 = rG_1 + 2sG_0. \tag{1.5}$$

Put $C_G = G_1^2 - rG_0G_1 - sG_0^2$. It is known that the terms of a binary recurrence $\{G\}$ and its associate sequence $\{H\}$ satisfy the equality

$$H_n^2 - DG_n^2 = 4C_G(-s)^n, \tag{1.6}$$

where $D = r^2 + 4s$.

2. Preparation and results

By (1.3) it follows that the coefficients of the investigated linear recurrences are $r = \kappa$ and $s = -1$. Thus $D = \kappa^2 - 4$, moreover

$$C_a = a_1^2 - ra_0a_1 - sa_0^2 = (\kappa + 2)q^2$$

and

$$C_b = b_1^2 - rb_0b_1 - sb_0^2 = ((p-3)^2 - \kappa(p-3) + 1)q^2.$$

Now we fix a mosaic given by $\{\tilde{p}, \tilde{q}\} = \{4, 5\}$. Then $\tilde{\kappa} = 4$, $\tilde{a}_n = 4\tilde{a}_{n-1} - \tilde{a}_{n-2}$, $\tilde{a}_1 = 5$, $\tilde{a}_2 = 25$, and $\tilde{b}_n = 4\tilde{b}_{n-1} - \tilde{b}_{n-2}$, $\tilde{b}_1 = 5$, $\tilde{b}_2 = 15$, moreover $\tilde{D} = 12$. The first ten terms of the sequences are given by the following table.

i	1	2	3	4	5	6	7	8	9	10
$\tilde{a}_i$	5	25	95	355	1325	4945	18455	68875	257045	959305
$\tilde{b}_i$	5	15	55	205	765	2855	10655	39765	148405	553855

Table 1: *Numbers of leaves and roots on level i connected with mosaic $\{4, 5\}$*

The associate sequences of $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ satisfy

$$\tilde{A}_n = 4\tilde{A}_{n-1} - \tilde{A}_{n-2} \quad \text{with} \quad \tilde{A}_1 = 30,\ \tilde{A}_2 = 90, \tag{2.1}$$

$$\tilde{B}_n = 4\tilde{B}_{n-1} - \tilde{B}_{n-2} \quad \text{with} \quad \tilde{B}_1 = 10,\ \tilde{B}_2 = 50, \tag{2.2}$$

respectively. Since $C_{\tilde{a}} = 150$, $C_{\tilde{b}} = -50$, by (1.6) we obtain the identities

$$\tilde{A}_n^2 - 12\tilde{a}_n^2 = 600 \quad \text{and} \quad \tilde{B}_n^2 - 12\tilde{b}_n^2 = -200. \tag{2.3}$$

In this paper, we target to solve

I. the diophantine equation $a_k = \tilde{a}_\ell$ in k and ℓ for certain mosaics $\{p, q\}$ (Section 3); further

II. the equations $a_\varepsilon = \tilde{a}_\ell$ in ℓ if $\varepsilon \in \{1, 2, 3\}$ and one of p and q is fixed (Section 4 and 5).

For the sequence $\{b_n\}$ analogous problems are examined.

The first question leads to simultaneous Pellian equations. The second problem requires different approaches depending on ε and the sequence $\{a_n\}$ (or $\{b_n\}$).

The observations are contained in the following theorems and Result 2.2. We always assume that

$$\{p, q\} \neq \{4, 4\}, \{4, 5\}.$$

Theorem 2.1. (1) *Let* $4 \leq p \leq 25$ *and* $4 \leq q \leq 18$. *Then the equation* $a_k = \tilde{a}_\ell$ *has only the trivial solution* $a_1 = \tilde{a}_1 = 5$ *for* $q = 5$ *and any* p.

(2) *If* $4 \leq p, q \leq 10$, *or* $11 \leq p \leq 25$ *and* $4 \leq q \leq 8$, *then the equation* $b_k = \tilde{b}_\ell$ *possesses only the solutions*

- $\{p, q\} = \{6, 5\}$, $b_1 = \tilde{b}_2 = 15$,
- $\{p, q\} = \{10, 5\}$, $b_2 = \tilde{b}_5 = 765$,
- $\{p, q\} = \{14, 5\}$, $b_1 = \tilde{b}_3 = 55$.

Result 2.2. (1) *If* $4 \leq p \leq 1\,600$, *then* $a_2 = \tilde{a}_\ell$ *is satisfied by*

- $\{p, q\} = \{26, 5\}$, $a_2 = \tilde{a}_4 = 335$,
- $\{p, q\} = \{90, 29\}$, $a_2 = \tilde{a}_8 = 68\,875$,
- $\{p, q\} = \{332, 5\}$, $a_2 = \tilde{a}_6 = 4\,945$,

(2) *In case of* $4 \leq q \leq 10\,000$, $a_3 = \tilde{a}_\ell$ *has no non-trivial small solution (i.e.* $p \leq 10\,000$*).*

(3) *Assume* $4 \leq p \leq 10\,000$ *or* $4 \leq q \leq 2\,800$. *Then* $\{p, q\} = \{10, 5\}$, $b_2 = \tilde{b}_5 = 765$ *satisfy the equation* $b_2 = \tilde{b}_\ell$.

Theorem 2.3. (1) *All the solutions to* $a_2 = \tilde{a}_\ell$, *with* $5 \leq q \leq 25$ *are given by*

- $q = 5$, $\ell = 2 + 2t$ $(t \in \mathbb{N}^+)$,
- $q = 19$, $\ell = 58 + 90t$ *and* $\ell = 78 + 90t$ $(t \in \mathbb{N})$,
- $q = 23$, $\ell = 28 + 88t$ $(t \in \mathbb{N})$,
- $q = 25$, $\ell = 32 + 33t$ $(t \in \mathbb{N})$.

(2) *All the solutions to* $b_1 = \tilde{b}_\ell$, *with* $5 \leq q \leq 25$ *are given by*

- $q = 9$, $\ell = 5 + 18t$ *and* $\ell = 14 + 18t$ $(t \in \mathbb{N})$,
- $q = 11$, $\ell = 3 + 10t$ *and* $\ell = 8 + 10t$ $(t \in \mathbb{N})$,
- $q = 15$, $\ell = 2 + 6t$ *and* $\ell = 5 + 6t$ $(t \in \mathbb{N})$,
- $q = 17$, $\ell = 5 + 18t$ *and* $\ell = 14 + 18t$ $(t \in \mathbb{N})$.

3. Type I: $a_k = \tilde{a}_\ell$ and $b_k = \tilde{b}_\ell$ with certain p and q (Proof of Theorem 2.1)

It is known that the binary recurrence sequences are periodic modulo any positive integer. A simple consideration shows that the terms $\tilde{a}_n$ are never divisible by 2, 3, 7, 11, 13, 17 (primes up to 25), while $\tilde{b}_n$ are never a multiple of 2, 7, 13, 19, 23 (primes also up to 25). On the other hand, $q \mid a_n$ and $q \mid b_n$ hold for any n. Consequently, there is no solution to the equation $a_k = \tilde{a}_\ell$ unless $q = 5, 19, 23, 25$. Indeed, by $q \mid a_n$, one needs only to check one period of $\{\tilde{a}_n\}$ modulo q. Similarly, $b_k = \tilde{b}_\ell$ may possess solution only when $q = 5, 9, 11, 15, 17, 25$. Unfortunately, we could achive the computations only for $q = 5$ regarded to $a_k = \tilde{a}_\ell$, and for $q = 5$ and $q = 9$ regarded to $b_k = \tilde{b}_\ell$ since the time demand of evaluation of the algorithm decribed below seemed to be too much for larger q values.

Suppose that p and q are given, and consider $a_k = \tilde{a}_\ell$. Assume that $x = a_k$ satisfies this equation. Then, by (1.6)

$$y^2 - (\kappa^2 - 4)x^2 = 4(\kappa + 2)q^2 \tag{3.1}$$

holds for some positive integer y. On the other hand, in the virtue of (2.3) (the source of (2.3) is (1.6)), $x = \tilde{a}_\ell$ is also a zero of the equation

$$z^2 - 12x^2 = 600 \tag{3.2}$$

for some positive suitable integer z. Clearly, (3.1) and (3.2) form a system of simultaneous Pellian equations. The `PellianSystem()` procedure, developed in [6] and implemented in MAGMA [5] is able to solve such a system if the coefficients are not too large.

If we take $b_k = \tilde{b}_\ell$, then (3.1) and (3.2) must be replaced by

$$y^2 - (\kappa^2 - 4)x^2 = 4((p-3)^2 - \kappa(p-3) + 1)q^2 \tag{3.3}$$

and

$$z^2 - 12x^2 = -200, \tag{3.4}$$

respectively.

We have checked the solutions of the appropriate system of Pellian equations by MAGMA, and the result of the computations is reported in Theorem 2.1.

To illustrate the time demand of the computations, we note that the MAGMA server needed approximately 21 days to show that $b_k = \tilde{b}_\ell$ has no solution in the case $\{p, q\} = \{8, 9\}$ (this was the worst case we considered).

4. Type II: $a_\varepsilon = \tilde{a}_\ell$, $b_\varepsilon = \tilde{b}_\ell$, part 1. (Background behind Result 2.2)

This section is devoted to deal with the equations above in the specific cases

1. $a_2 = \tilde{a}_\ell$, when parameter p of $\{a_n\}$ is fixed in the range $4 \le p \le 1\,600$,
2. $a_3 = \tilde{a}_\ell$, when parameter q of $\{a_n\}$ satisfies $4 \le q \le 10\,000$,
3. $b_2 = \tilde{b}_\ell$, when $p \in [4; 10\,000]$,
4. $b_2 = \tilde{b}_\ell$, when $q \in [4; 2\,800]$.

The common background behind the four problems is that all of them are linked to hyperelliptic diophantine equations of degree four. Observe, that a_2 and b_2 is a quadratic polynomial in q, similarly a_3 and b_2 has degree two in p.

Consider first

$$a_2 = \tilde{a}_\ell$$

with fixed p. Then, by the first identity of (2.3), a_2 satisfies

$$y^2 - 12a_2^2 = 600,$$

where $a_2 = f(q) = (\kappa + 1)q$ is a quadratic polynomial of q. Consequently we need to solve the quartic hyperelliptic equation

$$y^2 = 12f^2(q) + 600. \tag{4.1}$$

We use the `IntegralQuarticPoints()` procedure of MAGMA package to handle (4.1). Note that if the constant term of the polynomial on the right hand side of (4.1) is not a full square, then the procedure requires a solution (as input) to the equation to determine all solutions. In this case we scanned the interval $J = [-10\,000; 10\,000]$ for q to find a solution. It might occur that there is a solution outside J and not inside J, but we found no example to this.

If once we have determined a q, then we search back the corresponding subscript ℓ.

The analogy to the other 3 cases of this section is obvious: in the right hand side of (4.1) the polyomial f is being replaced by $f(p) = (\kappa^2 + \kappa - 1)q$, $f(q) = (\kappa(p-3) - 1)q$ and $f(p) = (\kappa(p-3) - 1)q$, respectively.

Solutions we found are listed in Result 2.2 (the list might be not full in accordance with the basic interval J which was used for finding a solution).

5. Type III: $a_\varepsilon = \tilde{a}_\ell$, $b_\varepsilon = \tilde{b}_\ell$, part 2. (Proof of Theorem 2.3)

Here we study the title equation in a few cases with small ε, which differ from the previous section. Recall that both of the sequences $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ are purely periodic for any positive integer modulus.

Since $a_1 = q$ the equation $a_1 = \tilde{a}_\ell$ has, trivially, infinitely many solutions.

The next problem is $a_2 = \tilde{a}_\ell$ with fixed q. (The case with fixed p has already been studied in Section 4.) Recall that $a_2 = (\kappa + 1)q$, more precisely

$$a_2 = q(q-2)(p-2) - q$$

is linear in p. Therefore we need to determine the common terms of an arithmetic progression and the sequence $\{\tilde{a}_n\}$. The situation does not change if we consider $b_1 = \tilde{b}_\ell$ with either fixed p or fixed q. Indeed, $b_1 = (p-3)q$ is linear both in p and q.

Obviously, $a_2 \equiv -q \pmod{q(q-2)}$. Consequently, the equation $a_2 = \tilde{a}_\ell$ is soluble if and only if we find at least one element in the sequence $\{\tilde{a}_n\}$, which is congruent $-q$ modulo $q(q-2)$. Because of the periodicity, one must check only one period of $\{\tilde{a}_n\}$ modulo $q(q-2)$.

Assume first that $q = 5$. Then for the modulus $q(q-2) = 15$ we have $\tilde{a}_{2+2t} \equiv -5$ (the cycle's length is 2, and $t \in \mathbb{N}$). Hence $a_2 = \tilde{a}_{2+2t}$, further

$$p = \frac{\tilde{a}_{2+2t} + q}{q(q-2)} + 2.$$

t	0	1	2	3	4	5
$a_2 = \tilde{a}_{2+2t}$	25	355	4945	68875	959305	13361395
p	4	26	332	4594	63956	890762

Table 2: *First few solutions to $a_2 = \tilde{a}_\ell$ when $q = 5$*

The first six t values yield the following solutions. (If $t = 0$ then the two sequences $\{a_n\}$ and $\{\tilde{a}_n\}$ coincide.)

If $q > 5$ the first non-trivial solution is occurred when $q = 19$. Here the length of the cycle is 90, and $q(q-2) \mid \tilde{a}_{58} + 19$, $q(q-2) \mid \tilde{a}_{78} + 19$. That is $a_2 = \tilde{a}_{58+90t}$ and $a_2 = \tilde{a}_{78+90t}$ ($t \in \mathbb{N}$) provide all solutions for suitable values p. For instance, $t = 0$ gives

$$p = 8\,437\,940\,669\,128\,098\,583\,408\,551\,589\,590$$

and

$$p = 2\,318\,394\,927\,973\,629\,460\,854\,893\,981\,169\,574\,319\,067\,870,$$

respectively.

The treatment is similar for $b_1 = \tilde{b}_\ell$. If $q = 5$, then solution always exists since $b_1 = (p-3)q$, $5 \mid \tilde{b}_\ell$, therefore $p = \tilde{b}_\ell/5 + 3$. ($\tilde{b}_2$ and $\tilde{b}_3$ give back solutions have already been appeared in Theorem 2.1.) Now $b_1 \equiv 0 \pmod q$, and fixing $q \geq 6$ the first solution appears for $q = 9$, when the cycle length is 18 (modulo q), and we have $b_1 = \tilde{b}_{5+18t}$ and $b_1 = \tilde{b}_{14+18t}$ ($t \in \mathbb{N}$). These results can be directly converted the results corresponding to p, therefore we omit the appropriate analysis.

The results we obtained are summarized in Theorem 2.3.

Finally, we examine the equation $a_3 = \tilde{a}_\ell$ with fixed q, further $b_3 = \tilde{b}_\ell$ when exactly one of p and q is given. In each case we have a polynomial of degree three, let say $\phi(x)$, and we look for the common values of the polynomial and a binary recurrence. By (1.6), the problem leads to the hyperelliptic equation

$$y^2 = 12\phi^2(x) + c$$

of degree 6, where the constant c is either 600 or -200. Since the leading coefficient on the right hand side is not a square, there is no genearal algorithm to solve. For example, $p = 5$ provides now

$$y^2 = 12q^2(9q^2 - 45q + 55)^2 + 600.$$

After dividing by 4, we have

$$y_1^2 = 243q^6 - 2430q^5 + 9045q^4 - 14850q^3 + 9075q^2 + 150,$$

and the techique of the solution is not known.

Acknowledgements. The authors thank P. Olajos for his valuable help in using MAGMA package.

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