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Research Article

Linear vector recursions of arbitrary order

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Abstract

Solution of various combinatorial problems often requires vector recurrences of higher order (i.e., the order is larger than 1). Assume that there are given matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_s$, all from $\mathbb{C}^{k\times k}$. These matrices allow us to define the vector recurrence $\bar{v}_n = \mathbf{A}_1\bar{v}_{n-1} + \mathbf{A}_2\bar{v}_{n-2} + \cdots + \mathbf{A}_s\bar{v}_{n-s}$ for the vectors $\bar{v}_n \in \mathbb{C}^k$, $n \geq s$. The paramount result of this paper is that we could separate the component sequences of the vectors and find a common linear recurrence relation to describe them. The principal advantage of our approach is a uniform treatment and the possibility of automatism. We could apply the main result to answer a problem that arose concerning the rows of the modified hyperbolic Pascal triangle with parameters $\{4,5\}$. We also verified two other statements from the literature in order to illustrate the power of the method.

Keywords: linear recurrence; vector recurrence; hyperbolic Pascal triangle; bounded anchored permutation; Fibonacci sequence.

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1. Introduction

The theory of linear recurrences often plays an important role in combinatorics and number theory. This paper describes a method to handle the problem of vector recurrences in general. We use this approach to solve a new question related to hyperbolic Pascal triangles (in short, HPT). Moreover, we illustrate the applicability of the method by checking the solutions of two problems appearing in the literature. The first one is associated with the number of bounded anchored permutations and the second one deals with combined Fibonacci sequences.

Vector recurrences

Let $s \geq 1$ and $k \geq 2$ denote two positive integers. Assume that there are given the matrices $\mathbf{A}_t = [a_{i,j}^{(t)}] \in \mathbb{C}^{k\times k}$ for $t = 1, 2, \ldots, s$. We define the vector recurrence

$$\bar{v}_n = \mathbf{A}_1\bar{v}_{n-1} + \mathbf{A}_2\bar{v}_{n-2} + \cdots + \mathbf{A}_s\bar{v}_{n-s}, \qquad n \geq s \tag{1}$$

with initial column vectors $\bar{v}_t = [v_1^{(t)}, v_2^{(t)}, \ldots, v_k^{(t)}]^\top \in \mathbb{C}^k$, $t = 0, 1, \ldots, s-1$.

We have developed a procedure in order to separate the component sequences $(v_i^{(t)})_{t\geq 0}$ for $i = 1, 2, \ldots, k$, and to give their own common recursive relation. First, we needed this method to solve a problem associated with the HPT given by the parameters $\{4,5\}$, but later it turned out that this approach might be very useful in the resolution of other questions. A fast search revealed a couple of problems related to (1), and later we chose two of them for demonstration.

The simplest version of (1) is

$$\bar{v}_n = \mathbf{A}_1\bar{v}_{n-1}, \tag{2}$$

when we have only one matrix, i.e., the order of vector recurrence is 1. Hence, the combination of component recurrences goes back only to one term. In this case, the characteristic polynomial of the matrix $\mathbf{A}_1$ implies a recurrence rule of order k valid for any of the separated component sequences of the vectors $\bar{v}_n$ (see, for example, [5, Lemma 2.1], or [4]). (Note that the basic field in [5, Lemma 2.1] is $\mathbb{R}$, but the statement can be extended without changes for the field $\mathbb{C}$ as well.) If one finds the zeros of the characteristic polynomial, then the zeros, together with the initial values, can provide an explicit form for each component sequence. The difficulty that may arise in practice is finding the zeros precisely. For the non-homogeneous version of (2), see [2] (with two sequences) and [5] (generally).

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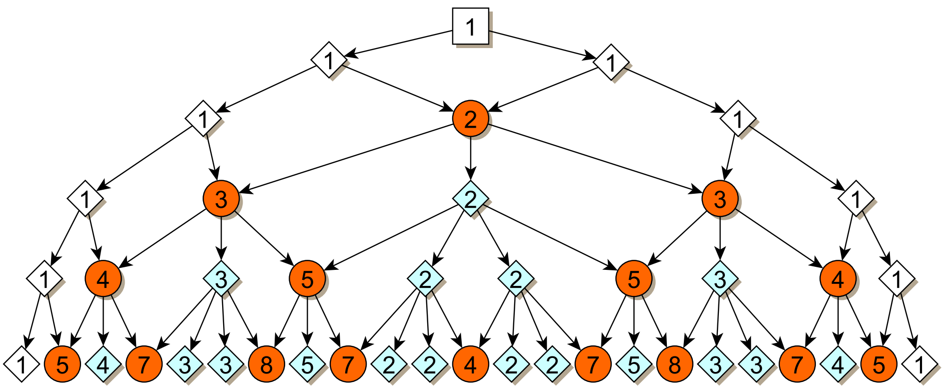

Figure 1.1: Hyperbolic Pascal triangle with parameters $p = 4$ and $q = 5$.

The crucial point of our approach is to transform (1) in order to obtain a corresponding vector recurrence in the form (2) (which will be given later in (6)). Section 2 is devoted to describing and clarifying the details.

A combinatorial problem in modified HPT$\{4, 5\}$

The notion of the hyperbolic Pascal triangle (see [2]) was introduced a few years ago as a variation of Pascal's arithmetic triangle. The novelty was to extend the link between the infinite graph corresponding to the regular square Euclidean mosaic and classical Pascal's triangle to the hyperbolic plane, which contains infinitely many types of regular mosaics. A hyperbolic Pascal triangle is characterized by the Schäfli's symbol $\{p, q\}$ according to the type of the regular mosaic in the hyperbolic plane. Now let $p = 4$, $q = 5$, which means that the regular polygon in the mosaic is square, and exactly 5 squares meet at each vertex. As an illustration, Figure 1.1 shows the first few rows of HPT$\{4, 5\}$, which can also be considered as a digraph.

In each row of this graph, apart from the leftmost and rightmost vertices (said to be *wingers*) there are two types of vertices. Some of the vertices (called *type* A for convenience, figured by red circles) have 2 ascendants and 3 descendants, the others (*types* B, illustrated by cyan rhombuses) have 1 ascendant and 4 descendants. The value of a vertex is the sum of the values of the ascendants. It gives the number of the shortest paths to the vertex from the basic vertex $\boxed{1}$ located on the top of the figure.

Belbachir et al. [2] discovered several properties of HPT's, we recall here only one result. The sum of the elements in row n gives the number of shortest paths ν_n from the base vertex to row n. We found that sequence $(\nu_n)_{n\geq 0}$ satisfies the ternary recurrence relation

$$\nu_n = 5\nu_{n-1} - 6\nu_{n-2} + 2\nu_{n-3}, \qquad n \geq 3,$$

the initial values are $\nu_0 = 1$, $\nu_1 = 2$, and $\nu_2 = 4$. This is sequence A087161 in [7]. In fact, we presented a more general statement in [2] associated with HPT$\{4, q\}$.

Now we modify HPT$\{4, 5\}$ as follows.

- Insert a new edge with double length from each vertex V having type A of row n to each accessible vertex (in the digraph, from V) of type B in row $(n + 2)$. (These edges are indicated by blue arrows in Figure 1.2.)
- Insert a new edge with double length from each vertex V having type B of row n to each accessible vertex (in the digraph, from V) of type A in row $(n + 2)$. (These edges are indicated by green dashed arrows in Figure 1.2.)

Recall that these new edges have double length. Now we ask for the number of shortest paths from the base vertex $\boxed{1}$ to row n.

Clearly, the brief answer is the sum of the values of the vertices located in row n. We denote it by s_n. After studying the properties of vector recurrences, we will answer the above question more precisely.

2. Inhomogeneous vector recurrence

We will show how vector recurrence (1) leads equivalently to a vector recurrence of order 1 having type (2). Replacing $\mathbf{A}_1$ by M formally in (2), we will see that M has size $sk \times sk$, and we will construct M itself.

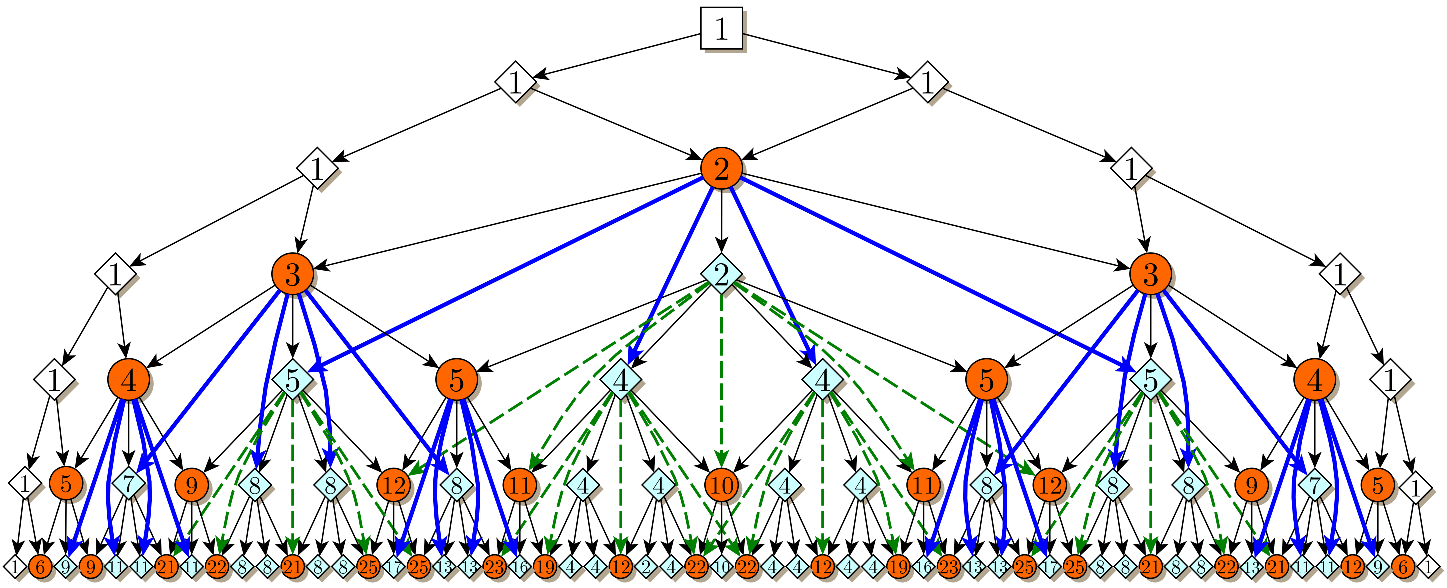

Figure 1.2: Modified hyperbolic Pascal triangle with parameters $p = 4$ and $q = 5$.

The ith ($i = 1, 2, \ldots, k$) component sequence $(v_i^{(t)})_{t\geq 0}$ can be given recursively from (1) by the formula

$$v_i^{(n)} = \sum_{j=1}^{k} a_{i,j}^{(1)} v_j^{(n-1)} + \sum_{j=1}^{k} a_{i,j}^{(2)} v_j^{(n-2)} + \cdots + \sum_{j=1}^{k} a_{i,j}^{(s)} v_j^{(n-s)} = \sum_{t=1}^{s}\sum_{j=1}^{k} a_{i,j}^{(t)} v_j^{(n-t)}. \tag{3}$$

The crucial point is to consider the vectors $\bar{v}_{n-1}, \bar{v}_{n-2}, \ldots, \bar{v}_{n-s}$ in (1) as the $(n-1)$th terms (i.e., vectors) of certain new vector sequences. More precisely, for $t = 1, 2, \ldots, s$ put

$$\bar{v}_{n-1}^{[t-1]} := \bar{v}_{n-t},$$

where the symbol $[t-1]$ indicates the $[t-1]$th new vector sequence. In particular, $t = 1$ returns with $\bar{v}_{n-1}^{[0]} = \bar{v}_{n-1}$. For the new component sequences, we introduce the notation

$$v_i^{[t-1,n-1]} := v_i^{(n-t)} \tag{4}$$

for all $i = 1, 2, \ldots k$ and for all $t = 1, 2, \ldots, s$. Applying it (3) becomes

$$v_i^{(n)} = v_i^{[0,n]} = \sum_{t=1}^{s}\sum_{j=1}^{k} a_{i,j}^{(t)} v_j^{[t-1,n-1]}.$$

Observe that we have

$$v_j^{[t-1,n]} = v_j^{n-t+1} = v_j^{[t-2,n-1]}. \tag{5}$$

These arguments and the notation above lead to the vector recurrence

$$\bar{V}_n = \mathbf{M}\bar{V}_{n-1} \tag{6}$$

analogous to (2), where

$$\bar{V}_n = \left[v_1^{[0,n]}, \ldots, v_k^{[0,n]}; v_1^{[1,n]}, \ldots, v_k^{[1,n]}; \ldots; v_1^{[s-1,n]}, \ldots, v_k^{[s-1,n]}\right]^\top \in \mathbb{C}^{sk}.$$

Furthermore, by equalities (4) and (5) matrix M can be given in the block matrix form

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{s-2} & \mathbf{A}_{s-1} & \mathbf{A}_s \\ \mathbf{I}_k & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{I}_k & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{I}_k & \mathbf{O} \end{bmatrix} \in \mathbb{C}^{sk\times sk},$$

where $\mathbf{O}$ is the $k \times k$ zero matrix, $\mathbf{I}_k$ is the $k \times k$ identity matrix. According to the results of [5, Lemma 2.1] we need to determine the characteristic polynomial

$$c_{\mathbf{M}}(x) = \det(x\mathbf{I}_{sk} - \mathbf{M}) = (-1)^{sk} \det(\mathbf{M} - x\mathbf{I}_{sk}) \tag{7}$$

of matrix M in order to obtain usual recurrence rule for the sequences $(v_i^{[t,n]})_n$.

Obviously,

$$\det(\mathbf{M} - x\mathbf{I}_{sk}) = \begin{vmatrix} \mathbf{A}_1 - x\mathbf{I}_k & \mathbf{A}_2 & \cdots & \mathbf{A}_{s-2} & \mathbf{A}_{s-1} & \mathbf{A}_s \\ \mathbf{I}_k & -x\mathbf{I}_k & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{I}_k & -x\mathbf{I}_k & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{I}_k & -x\mathbf{I}_k \end{vmatrix}.$$

Using $(s-1)k$ suitable elementary column transformations as follows, one obtains an equivalent determinant whose value is easy to find. Taking the columns one by one from the first column to the $(s-1)k$th we execute the transformations

$$\text{column}_{k+j}^{new} = \text{column}_{k+j}^{old} + x \cdot \text{column}_j^{actual}$$

step by step for $j = 1, 2, \ldots, (s-1)k$ in this order. This sequence of transformations results the equivalent determinant

$$\det(\mathbf{M} - x\mathbf{I}_{sk}) = \begin{vmatrix} \mathbf{A}_1^\star & \mathbf{A}_2^\star & \cdots & \mathbf{A}_{s-2}^\star & \mathbf{A}_{s-1}^\star & \mathbf{A}_s^\star \\ \mathbf{I}_k & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{I}_k & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{I}_k & \mathbf{O} \end{vmatrix},$$

where

$$\mathbf{A}_1^\star = \mathbf{A}_1 - x\mathbf{I}_k, \quad \mathbf{A}_2^\star = \mathbf{A}_2 + x\mathbf{A}_1^\star, \ \ldots, \ \mathbf{A}_s^\star = \mathbf{A}_s + x\mathbf{A}_{s-1}^\star. \tag{8}$$

The consecutive application of Laplace expansions for the rows $k+1, k+2, \ldots, sk$ provides

$$\det(\mathbf{M} - x\mathbf{I}_{sk}) = \left((-1)^k\right)^{(s-1)k} \det(\mathbf{A}_s^\star). \tag{9}$$

On the other hand, it follows from (8) that

$$\mathbf{A}_s^\star = \mathbf{A}_s + x\mathbf{A}_{s-1} + x^2\mathbf{A}_{s-2} + \cdots + x^{s-1}\mathbf{A}_1 - x^s\mathbf{I}_k.$$

Combining it with (7) and (9), finally we have just proved the following result:

Theorem 2.1. *The characteristic polynomial of matrix* M *is*

$$c_{\mathbf{M}}(x) = (-1)^{sk^2+sk-k^2} \det\left(\mathbf{A}_s + x\mathbf{A}_{s-1} + x^2\mathbf{A}_{s-2} + \cdots + x^{s-1}\mathbf{A}_1 - x^s\mathbf{I}_k\right).$$

Note that the coefficient $(-1)^{sk^2+sk-k^2}$ equals 1 if k is even; otherwise, it takes -1. But it has no influence on the zeros of the characteristic polynomial or on the recursive relations related to the component sequences $(v_i^{[t,n]})_{n\geq 0}$.

In practice, besides the initial values, we will need the zeros of $c_{\mathbf{M}}(x)$ in order to give explicit form for the component sequences, but this method is well known.

3. Solution to the modified HPT$\{4,5\}$ problem

Let a_n and b_n denote the sum of elements in row n having type A and B, respectively. Clearly, $s_n = a_n + b_n + 2$ holds for $n \geq 1$. Similarly to the proof of [2, Theorem 2], but according to the new edges we have

$$\begin{aligned} a_n &= 2a_{n-1} + 2b_{n-1} + 5b_{n-2} + 2, \\ b_n &= a_{n-1} + 2b_{n-1} + 4a_{n-2} \end{aligned}$$

if $n \geq 3$. Indeed, a vertex of type A has 4 descendants having type B two rows below, and an element of type B has 5 descendants having type A two rows below (see Figure 1.2). Put

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

n	1	2	3	4	5	6
a_n	0	2	6	18	84	416
b_n	0	0	2	18	78	312
s_n	2	4	10	38	164	730

Table 1: Initial values of sequences (a_n), (b_n), and (s_n).

Thus,

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \mathbf{A}_1 \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} + \mathbf{A}_2 \begin{bmatrix} a_{n-2} \\ b_{n-2} \end{bmatrix} + \bar{w}, \tag{10}$$

which is a specific inhomogenous version of (1). Decreasing all the subscripts by 1 in (10), and taking the difference of the two equalities, it leads to

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = (\mathbf{A}_1 + \mathbf{I}_2) \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} + (\mathbf{A}_2 - \mathbf{A}_1) \begin{bmatrix} a_{n-2} \\ b_{n-2} \end{bmatrix} + (-\mathbf{A}_2) \begin{bmatrix} a_{n-3} \\ b_{n-3} \end{bmatrix}.$$

This is exactly a vector recurrence of type (1) with $k = 2$, and $s = 3$. Now apply the method we described in the previous section. The characteristic polynomial of matrix M is

$$\begin{aligned} c_{\mathbf{M}}(x) &= x^6 - 6x^5 + 11x^4 - 21x^3 + 8x^2 + 27x - 20 \\ &= (x+1)(x-1)^2(x^3 - 5x^2 + 7x - 20). \end{aligned}$$

Hence the two sequences (a_n) and (b_n) both satisfy the recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 21x_{n-3} - 8x_{n-4} - 27x_{n-5} + 20x_{n-6}, \quad n \geq 7$$

of order 6. The initial values given in Table 1 can be found by Figure 1.2.

But this is not a minimal recurrence! The Berlekamp-Massey algorithm would provide that (see, for example, [8, p. 29-30]), but here we use a ”natural” way as follows. To find the minimal recurrence relation, we need an upper bound ℓ on its order, which is now clearly $\ell = 6$. Then the minimal recursion is uniquely determined by the first 2ℓ elements of the sequence because we need to get ℓ equations to solve for the unknown coefficients. The investigation returns with the minimal recursion

$$x_n = 5x_{n-1} - 6x_{n-2} + 15x_{n-3} + 7x_{n-4} - 20x_{n-5}, \tag{11}$$

which is valid for both (a_n) and (b_n). Note that it corresponds the polynomial $c_{\mathbf{M}}(x)/(x-1)$.

Finally, we show that the sum of elements sequence (s_n) also satisfies (11). We easily have

$$\begin{aligned} s_n &= a_n + b_n + 2 \\ &= 5(a_{n-1} + b_{n-1} + 2 - 2) - 6(a_{n-2} + b_{n-2} + 2 - 2) + \cdots - 20(a_{n-5} + b_{n-5} + 2 - 2) + 2 \\ &= 5s_{n-1} - 6s_{n-2} + 15s_{n-3} + 7s_{n-4} - 20s_{n-5}. \end{aligned}$$

Hence, we proved the following result:

Theorem 3.1. *The sum of the element sequence (s_n) in the modified HPT$\{4,5\}$ triangle satisfies the following linear recurrence of order 5, which is minimal:*

$$s_n = 5s_{n-1} - 6s_{n-2} + 15s_{n-3} + 7s_{n-4} - 20s_{n-5}, \quad n \geq 6.$$

At the end of this section, we remark that we could use our approach analogously for the modified HPT$\{4, q\}$, where $q \geq 5$ is an integer.

We also note that the explicit solution for the characteristic polynomial of matrix M can be obtained if the size of the matrices $\mathbf{A}_t$ is $k = 2$, see [6].

4. Examples

There exist combinatorial problems in the literature whose solution requires two or more combined simultaneous linear recurrences. Here we choose two solved questions with different flavors.

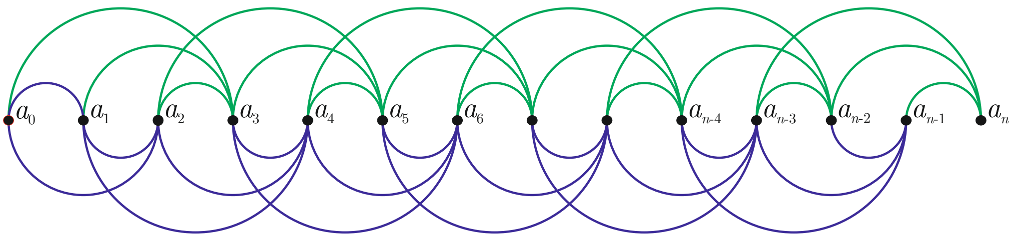

Figure 4.1: Graph $\mathcal{G}$ associated with 3-bounded anchored permutation problem.

1st example: 3-bounded anchored permutations

Sequence A249665 in The On-Line Encyclopedia of Integer Sequences [7] has the interpretation "*The number of permutations* p *of* $\{1, ..., n\}$ *such that* $p(1) = 1$*,* $p(n) = n$*, and* $|p(i)-p(i+1)|$ *is in* $\{1, 2, 3\}$ *for all* i *from 1 to* $n-1$*.*" The encyclopedia gives the following formula. Let A249665 be denoted by a_n. Moreover, let $a_1 = 1$, $g_1 = h_1 = 0$. For all $n < 1$, let $a_n = g_n = h_n = 0$. Then,

$$\begin{aligned} a_n &= a_{n-1} + g_{n-1} + h_{n-1}, \\ g_n &= a_{n-2} + a_{n-3} + a_{n-4} - a_{n-6} + g_{n-2} + g_{n-4} + h_{n-2}, \\ h_n &= 2a_{n-3} + 2a_{n-4} + a_{n-5} - a_{n-7} + g_{n-3} + g_{n-5} + h_{n-3}. \end{aligned}$$

The authors in [3], among many valuable arguments, for example, to find the previous system of equalities itself, gave the solution for the sequence (a_n) as a linear recurrence of order 8 in the form

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} + a_{n-4} + a_{n-5} - a_{n-7} - a_{n-8}.$$

Their method is based on the generating function technology, and on some specific observations linked to the sequences. The main advantage of our method is the automatism, and we present it now briefly.

Obviously, there exist matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_7 \in \mathbb{R}^{3\times 3}$ such that the vector recurrence

$$\begin{bmatrix} a_n \\ g_n \\ h_n \end{bmatrix} = \mathbf{A}_1 \begin{bmatrix} a_{n-1} \\ g_{n-1} \\ h_{n-1} \end{bmatrix} + \mathbf{A}_2 \begin{bmatrix} a_{n-2} \\ g_{n-2} \\ h_{n-2} \end{bmatrix} + \cdots + \mathbf{A}_7 \begin{bmatrix} a_{n-7} \\ g_{n-7} \\ h_{n-7} \end{bmatrix}$$

holds. For instance,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ etc.}$$

The matrix M has characteristic polynomial

$$\begin{aligned} c_{\mathbf{M}}(x) &= \begin{vmatrix} -x^7 + x^6 & x^6 & x^6 \\ x^5 + x^4 + x^3 - x & -x^7 + x^5 + x^3 & x^5 \\ 2x^4 + 2x^3 + x^2 - 1 & x^4 + x^2 & -x^7 + x^4 \end{vmatrix} \\ &= -x^{12}(x+1)(x^8 - 2x^7 + x^6 - 2x^5 - x^4 - x^3 + x + 1). \end{aligned}$$

Clearly, the factor $-x^{12}$ does not count. We focus again on the unique minimal recursion. Both the "natural" method and Berlekamp-Massey algorithm provide that it follows from the factor $x^8 - 2x^7 + x^6 - 2x^5 - x^4 - x^3 + x + 1$, which coincides to the result of [3].

Consider a simple graph $\mathcal{G}$ with vertex set $\{1, 2, \ldots, n\}$ such that there is an edge between u and v if and only if $|u-v| \le 3$ (see Figure 4.1).

The problem of 3-bounded anchored permutations is equivalent to finding the number of Hamiltonian paths in this graph from 1 to n. There is an amazing isomorphic spatial illustration of graph $\mathcal{G}$ as faces of regular tetrahedrons are glued consecutively together. This is called the Boerdijk-Coxeter tetrahelix graph given in Figure 4.2.

2nd example: combined Fibonacci sequences

Atanassov et al. wrote a couple of papers on the so-called combined Fibonacci sequences (see [1] and the references therein). Our method offers a standard procedure to handle problems having similarity to the question of [1].

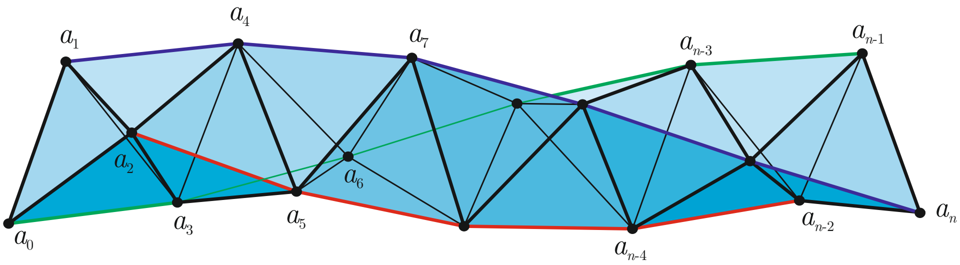

Figure 4.2: Boerdijk-Coxeter tetrahelix graph associated with 3-bounded anchored permutation problem.

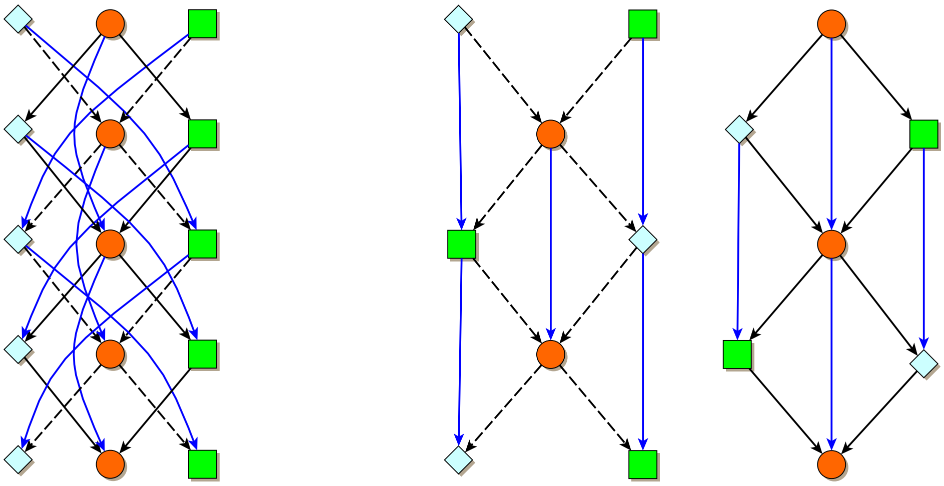

Figure 4.3: Graph illustration of Atanassov's combined Fibonacci problem.

Although the example is from [1], here we use our own notation. Let a, b, c, d, e, f be arbitrary real numbers. There are given two initial vectors and the coefficient matrices

$$\bar{v}_0 = \begin{bmatrix} 2a \\ b \\ 2c \end{bmatrix}, \quad \bar{v}_1 = \begin{bmatrix} 2d \\ e \\ 2f \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

respectively. Now $k = 3$, $s = 2$, and

$$\bar{v}_n = \mathbf{A}_1 \bar{v}_{n-1} + \mathbf{A}_2 \bar{v}_{n-2}, \qquad n \geq 2. \tag{12}$$

Note that the original sequence-wise description can be found in [1], where Figure 1 (of [1]) is understandable but not accurate enough in a graph-theoretical sense. Therefore we enclose Figure 4.3 in order to visualize the connections of the three sequences. The left-hand side of the figure shows the complete situation. Observe this graph is not connected, it is a union of two connected planar graphs (see the right side of the figure). (The weight of arrows is 1 apart from the dashed arrows having weight $1/2$. The three colors symbolize the three sequences.)

By Theorem 2.1 we gain

$$\det\left(\mathbf{A}_2 + x\mathbf{A}_1 - x^2\mathbf{I}_2\right) = \begin{vmatrix} -x^2 & x & 1 \\ \\ \frac{1}{2}x & -x^2+1 & \frac{1}{2}x \\ \\ 1 & x & -x^2 \end{vmatrix} = -(x^6 - 2x^4 - 2x^2 + 1),$$

which is factorized as $-(x^2+1)(x^2-x-1)(x^2+x-1)$. Since the zeros of this polynomial are $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $-\alpha$, $-\beta$, i, $-i$ therefore we have the formula

$$\omega_n = c_1\alpha^n + c_2\beta^n + c_3(-\alpha)^n + c_4(-\beta)^n + c_5 i^n + c_6(-i)^n, \quad n \geq 0, \tag{13}$$

where (ω_n) means any of the three component sequences of order six. Here the coefficients c_i are unknowns, but we can find them using the initial values $\omega_0, \omega_1, \ldots, \omega_5$.

Assume that we deal only with the first component sequence (it is denoted by (α_n) in [1]). Beside $\omega_0 = 2a$, $\omega_1 = 2d$ we determine $\omega_2 = 2c + e$, $\omega_3 = b + d + 3f$, $\omega_4 = 3a + c + 3e$, and $\omega_5 = 3b + 6d + 4f$ via (12). The system of 6 linear equations coming from (13) for $n = 0, 1, \ldots, 5$ has the solution

$$c_1 = \frac{1}{2\sqrt{5}}\left(\frac{a+b+c}{\alpha} + (d+e+f)\right), \qquad c_2 = -\frac{1}{2\sqrt{5}}\left(\frac{a+b+c}{\beta} + (d+e+f)\right),$$

$$c_3 = \frac{1}{2\sqrt{5}}\left(\frac{a-b+c}{\alpha} - (d-e+f)\right), \qquad c_4 = -\frac{1}{2\sqrt{5}}\left(\frac{a-b+c}{\beta} - (d-e+f)\right),$$

$$c_5 = \frac{1}{2}\left((a-c) - (d-f)i\right), \qquad c_6 = \frac{1}{2}\left((a-c) + (d-f)i\right).$$

We now have the explicit form for ω_n, but it has a favorable version if we eliminate the coefficients of $a, b, \ldots, f$, respectively. Here we consider only a, the other cases are left to the reader (for checking, see [1]). In (13), parameter a (via $c_1, c_2, \ldots, c_6$) has the coefficient

$$\frac{1}{2}\left(\frac{\alpha^{n-1} - \beta^{n-1} + (-1)^n\alpha^{n-1} - (-1)^n\beta^{n-1}}{\sqrt{5}} + i^n + (-i)^n\right) = \frac{1}{2}\left(F_{n-1} + (-1)^n F_{n-1} + i^n + (-i)^n\right).$$

This is zero if n is odd, and $F_{n-1} \pm 1$ otherwise. Note that it coincides perfectly with the result provided in [1]. Clearly, we could find similarly the coefficients of other parameters $b, c, \ldots, f$, too.

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