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FIBONACCI DIOPHANTINE TRIPLES

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Abstract. In this paper, we show that there are no three distinct positive integers a, b, c such that $ab+1$, $ac+1$, $bc+1$ are all three Fibonacci numbers.

1. Introduction

A *Diophantine m-tuple* is a set of $\{a_1, \ldots, a_m\}$ of positive rational numbers or integers such that a_ia_j+1 is a square for all $1 \le i < j \le m$. Diophantus found the rational quadruple $\{1/16, 33/16, 17/4, 105/16\}$, while Fermat found the integer quadruple $\{1, 3, 8, 120\}$. Infinitely many Diophantine quadruples of integers are known and it is conjectured that there is no Diophantine quintuples. This was almost proved by Dujella [5], who showed that there can be at most finitely many Diophantine quintuples and all of them are, at least in theory, effectively computable. In the rational case, it is not known that the size m of the Diophantine m-tuples must be bounded and a few examples with $m = 6$ are known by the work of Gibbs [8]. We also note that some generalization of this problem for squares replaced by higher powers (of fixed, or variable exponents) were treated by many authors (see [1, 2, 9, 13] and [10]).

In the paper [7], the following variant of this problem was treated. Let r and s be nonzero integers such that $\Delta = r^2 + 4s \neq 0$. Let $(u_n)_{n\ge 0}$ be a binary recurrence sequence of integers satisfying the recurrence

$$u_{n+2} = ru_{n+1} + su_n \qquad \text{for all } n \ge 0.$$

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It is well-known that if we write α and β for the two roots of the *characteristic equation* $x^2 - rx - s = 0$, then there exist constants $\gamma,\ \delta \in \mathbb{K} = \mathbb{Q}[\alpha]$ such that

$$u_n = \gamma\alpha^n + \delta\beta^n \qquad \text{holds for all } n \geq 0. \tag{1.1}$$

Assume further that the sequence $(u_n)_{n\geq 0}$ is *nondegenerate*, which means that $\gamma\delta \neq 0$ and α/β is not root of unity. We shall also make the convention that $|\alpha| \geq |\beta|$.

A Diophantine triple with values in the set $\mathcal{U} = \{u_n : n \geq 0\}$ is a set of three distinct positive integers $\{a, b, c\}$ such that $ab+1,\ ac+1,\ bc+1$ are all in $\mathcal{U}$. Note that if $u_n = 2^n + 1$ for all $n \geq 0$, then there are infinitely many such triples (namely, take $a,\ b,\ c$ to be any distinct powers of two). The main result in [7] shows that the above example is representative for the sequences $(u_n)_{n\geq 0}$ with real roots for which there exist infinitely many Diophantine triples with values in $\mathcal{U}$. The precise result proved there is the following.

Theorem 1.1. *Assume that $(u_n)_{n\geq 0}$ is a nondegenerate binary recurrence sequence with $\Delta > 0$ such that there exist infinitely many sextuples of nonnegative integers $(a, b, c; x, y, z)$ with $1 \leq a < b < c$ such that*

$$ab + 1 = u_x, \qquad ac + 1 = u_y, \qquad bc + 1 = u_z. \tag{1.2}$$

Then $\beta \in \{\pm 1\}$, $\delta \in \{\pm 1\}$, $\alpha,\ \gamma \in \mathbb{Z}$. Furthermore, for all but finitely many of the sextuples $(a, b, c; x, y, z)$ as above one has $\delta\beta^z = \delta\beta^y = 1$ and one of the following holds:

(i) $\delta\beta^x = 1$. In this case, one of δ or $\delta\alpha$ is a perfect square;
(ii) $\delta\beta^x = -1$. In this case, $x \in \{0, 1\}$.

No finiteness result was proved for the case when $\Delta < 0$. The case $\delta\beta^z = 1$ is not hard to handle. When $\delta\beta^z \neq 1$, results from Diophantine approximations relying on the Subspace Theorem, as well as on the finiteness of the number of solutions of nondegenerate unit equations with variables in a finitely generated multiplicative group and bounds for the greatest common divisor of values of rational functions at units points in the number fields setting, allow one to reduce the problem to elementary considerations concerning polynomials.

The Fibonacci sequence $(F_n)_{n\geq 0}$ is the binary recurrent sequence given by $(r, s) = (1, 1)$, $F_0 = 0$ and $F_1 = 1$. It has $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. According to Theorem 1.1, there should be only finitely many triples of distinct positive integers $\{a, b, c\}$ such that $ab+1,\ ac+1,\ bc+1$ are all three Fibonacci numbers. Our main result here is that in fact there are no such triples.

Theorem 1.2. *There do not exist positive integers $a < b < c$ such that*

$$ab + 1 = F_x, \qquad ac + 1 = F_y, \qquad bc + 1 = F_z, \tag{1.3}$$

where $x < y < z$ are positive integers.

Let us remark that since the values $n = 1,\ 2,\ 3$ and 5 are the only positive integers n such that $F_n = k^2 + 1$ holds with some suitable integer k (see [6]), it follows from Theorem 1.2 that all the solutions of equation (2.1) under the more relaxed condition $0 < a \leq b \leq c$ are

$$(a,b,c;x,y,z) = \begin{cases} (1,1,F_t-1;3,t,t), & t \geq 3; \\ (2,2,(F_t-1)/2;5,t,t), & t \geq 4,\ t \not\equiv 0 \pmod 3; \end{cases}$$

Note also that there are at least two *rational* solutions $0 < a < b < c$, namely

$$(a,b,c;x,y,z) = (2/3,3,18;4,7,10)\,,\ (9/2,22/3,12;9,10,11)\,.$$

It would be interesting to decide whether equation (1.3) has only finitely many rational solutions $(a,b,c;x,y,z)$ with $0 < a < b < c$, and in the affirmative case whether the above two are the only ones.

2. Proof of Theorem 1.2

2.1. *Preliminary results.* In the sequel, we suppose that $1 \leq a < b < c$ and $4 \leq x < y < z$. We write $(L_n)_{n \geq 0}$ for the companion sequence of the Fibonacci numbers given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. It is well-known (see, for example, Ron Knott's excellent web-site on Fibonacci numbers [11], or Koshy's monograph [12]), that the formulae

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n$$

hold for all $n \geq 0$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

We shall need the following statements.

Lemma 2.1. *The following divisibilities hold:*

(i) $\gcd(F_u, F_v) = F_{\gcd(u,v)}$;

(ii) $\gcd(L_u, L_v) = \begin{cases} L_{\gcd(u,v)}, & \text{if } \frac{u}{\gcd(u,v)} \equiv \frac{v}{\gcd(u,v)} \equiv 1 \pmod 2; \\ 1 \text{ or } 2, & \text{otherwise;} \end{cases}$

(iii) $\gcd(F_u, L_v) = \begin{cases} L_{\gcd(u,v)}, & \text{if } \frac{u}{\gcd(u,v)} \not\equiv \frac{v}{\gcd(u,v)} \equiv 1 \pmod 2; \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$

Proof. This is well-known (see, for instance [3, proof of Theorem VII.]). ◻

Lemma 2.2. *The following formulae hold:*

$$F_u - 1 = \begin{cases} F_{\frac{u-1}{2}}\, L_{\frac{u+1}{2}}, & \text{if } u \equiv 1 \pmod 4; \\ F_{\frac{u+1}{2}}\, L_{\frac{u-1}{2}}, & \text{if } u \equiv 3 \pmod 4; \\ F_{\frac{u-2}{2}}\, L_{\frac{u+2}{2}}, & \text{if } u \equiv 2 \pmod 4; \\ F_{\frac{u+2}{2}}\, L_{\frac{u-2}{2}}, & \text{if } u \equiv 0 \pmod 4. \end{cases}$$

Proof. This too is well-known (see, for example [14, Lemma 2]). ◻

Lemma 2.3. *Let u_0 be a positive integer. Put*

$$\varepsilon_i = \log_\alpha \left(1 + (-1)^{i-1}\left(\frac{|\beta|}{\alpha}\right)^{u_0}\right), \qquad \delta_i = \log_\alpha \left(\frac{1 + (-1)^{i-1}\left(\frac{|\beta|}{\alpha}\right)^{u_0}}{\sqrt{5}}\right)$$

for $i = 1,\ 2$, respectively. Here, $\log_\alpha$ is the logarithm in base α. Then for all integers $u \geq u_0$, the two inequalities

$$\alpha^{u+\varepsilon_2} \leq L_u \leq \alpha^{u+\varepsilon_1} \tag{2.1}$$

and

$$\alpha^{u+\delta_2} \leq F_u \leq \alpha^{u+\delta_1} \tag{2.2}$$

hold.

Proof. Let $c_0 = 1$, or $\sqrt{5}$, according to whether $u_n = L_n$ or $u_n = F_n$, respectively. Obviously,

$$\left.\begin{array}{l} L_u \\ F_u \end{array}\right\} \leq \frac{\alpha^u + |\beta|^{u_0}}{c_0} \leq \frac{\alpha^u\left(1 + \frac{|\beta|^{u_0}}{\alpha^u}\right)}{c_0} \leq \alpha^u \left(\frac{1 + \left(\frac{|\beta|}{\alpha}\right)^{u_0}}{c_0}\right),$$

which proves the upper bounds from the formulae (2.1) and (2.2), respectively. Similarly,

$$\left.\begin{array}{l} L_u \\ F_u \end{array}\right\} \geq \frac{\alpha^u - |\beta|^{u_0}}{c_0} \geq \frac{\alpha^u\left(1 - \frac{|\beta|^{u_0}}{\alpha^u}\right)}{c_0} \geq \alpha^u \left(\frac{1 - \left(\frac{|\beta|}{\alpha}\right)^{u_0}}{c_0}\right)$$

lead to the lower bounds from the formulae (2.1) and (2.2), respectively. □

Lemma 2.4. *Suppose that $a > 0$ and $b \geq 0$ are real numbers, and that u_0 is a positive integer. Then for all integers $u \geq u_0$, the inequality*

$$a\alpha^u + b \leq \alpha^{u+\kappa}$$

holds, where $\kappa = \log_\alpha\left(a + \frac{b}{\alpha^{u_0}}\right)$.

Proof. This is obvious. □

Lemma 2.5. *Assume that $a,\ b,\ z$ are integers. Furthermore, suppose that all the expressions appearing inside the* gcd*'s below are also integers. Then the following hold:*

(i) If $a \neq b$, then $\gcd\left(\frac{z+a}{2}, \frac{z+b}{4}\right) \leq \left|\frac{a-b}{2}\right|$. Otherwise, $\gcd\left(\frac{z+a}{2}, \frac{z+b}{4}\right) = \frac{z+b}{4}$;

(ii) If $3a \neq b$, then $\gcd\left(\frac{z+a}{2}, \frac{3z+b}{8}\right) \leq \left|\frac{3a-b}{2}\right|$. Otherwise, $\gcd\left(\frac{z+a}{2}, \frac{3z+b}{8}\right) = \frac{z+a}{8}$.

Proof. This is an easy application of the Euclidean algorithm. □

Lemma 2.6. *Assume that $z \geq 8$ is an integer. Then the following hold:*

(i) If z is odd, then $L_{\frac{z-1}{2}} < \sqrt{2F_z}$;
(ii) If z is even, then $L_{\frac{z-2}{2}} < \sqrt{F_z}$.

Proof. For (i), note that

$$L^2_{\frac{z-1}{2}} = L_{z-1} + 2(-1)^{z-1} \le L_{z-1} + 2 = F_{z-2} + F_z + 2,$$

and the right hand side above is easily seen to be smaller than $2F_z$ when $z \ge 8$. For (ii), we similarly have

$$L^2_{\frac{z-2}{2}} \le L_{z-2} + 2 = F_{z-3} + F_{z-1} + 2 < F_z,$$

where the last inequality is equivalent to $F_{z-3}+2 < F_{z-2}$, or $2 < F_{z-4}$, which is fulfilled for $z \ge 8$. ∎

Lemma 2.7. *All positive integer solutions of the system* (1.3) *satisfy* $z \le 2y$.

Proof. The last two equations of system (1.3) imply that c divides both $F_y - 1$ and $F_z - 1$. Consequently,

$$c \mid \gcd(F_y - 1, F_z - 1). \tag{2.3}$$

Obviously, $F_z = bc + 1 < c^2$; hence, $\sqrt{F_z} < c$. From (2.3), we obtain $\sqrt{F_z} < F_y$. Clearly,

$$\sqrt{\frac{\alpha^z - 1}{\sqrt{5}}} < \sqrt{F_z} < F_y < \frac{\alpha^y + 1}{\sqrt{5}}. \tag{2.4}$$

Since $y \ge 5$ entails $\alpha^y + 1 < \sqrt[4]{5}\,\alpha^y$, we get $\alpha^z - 1 < \alpha^{2y}$, which easily leads to the conclusion that $2y \ge z$. ∎

2.2. *The Proof of Theorem 1.2.* By Lemma 2.7, we have

$$\sqrt{F_z} < \gcd(F_z - 1, F_y - 1). \tag{2.5}$$

Applying Lemma 2.2, we obtain

$$\gcd(F_z - 1, F_y - 1) = \gcd\left(F_{\frac{z-i}{2}}\, L_{\frac{z+i}{2}}\ ,\ F_{\frac{y-j}{2}}\, L_{\frac{y+j}{2}}\right) \le \tag{2.6}$$

$$\le \gcd\left(F_{\frac{z-i}{2}}\ ,\ F_{\frac{y-j}{2}}\right) \gcd\left(F_{\frac{z-i}{2}}\ ,\ L_{\frac{y+j}{2}}\right) \gcd\left(L_{\frac{z+i}{2}}\ ,\ F_{\frac{y-j}{2}}\right) \gcd\left(L_{\frac{z+i}{2}}\ ,\ L_{\frac{y+j}{2}}\right),$$

where $i, j \in \{\pm1, \pm2\}$. The values i and j depend on the residue classes of z and y modulo 4, respectively. In what follows, we let $d_1,\ d_2,\ d_3$ and d_4 denote suitable positive integers which will be defined shortly.

Lemma 2.1 yields

$$m_1 = F_{\gcd\left(\frac{z-i}{2}, \frac{y-j}{2}\right)} = F_{\frac{z-i}{2d_1}}. \tag{2.7}$$

The second factor m_2 on the right hand of (2.6) can be 1, 2, or

$$m_2 = L_{\gcd\left(\frac{z-i}{2}, \frac{y+j}{2}\right)} = L_{\frac{z-i}{2d_2}}. \tag{2.8}$$

The third factor m_3 is again 1, 2, or

$$m_3 = L_{\gcd\left(\frac{z+i}{2}, \frac{y-j}{2}\right)} = L_{\frac{z+i}{2d_3}}. \tag{2.9}$$

Finally, if the fourth factor m_4 is neither 1 nor 2, then

$$m_4 = L_{\gcd\left(\frac{z+i}{2}, \frac{y+j}{2}\right)} = L_{\frac{z+i}{2d_4}}. \tag{2.10}$$

We now distinguish two cases.

CASE 1. $z \leq 150$.

In this case, we ran an exhaustive computer search to detect all positive integer solutions of system (1.3). Observe that we have

$$a = \sqrt{\frac{(F_x - 1)(F_y - 1)}{F_z - 1}}\,, \qquad 4 \leq x < y < z \leq 150.$$

Going through all the eligible values for $x,\ y$ and z, and checking if the above number a is an integer, we found no solution to system (1.3).

CASE 2. $z > 150$.

In this case, Lemma 2.3 gives $-2 < \delta_1$ for F_z. Hence, $\alpha^{\frac{z-2}{2}} < \sqrt{F_z}$. If $d_k \geq 5$ holds for all $k = 1,\ 2,\ 3,\ 4$, then the subscripts $\frac{z \pm i}{2d_k}$ of the Fibonacci and Lucas numbers appearing in (2.7)–(2.10) are at most $\frac{z \pm i}{10}$ each. Lemma 2.3 now gives that $\varepsilon_2 < 0.5$ and $\delta_2 < -1$ hold for $L_{\frac{z\pm i}{10}}$ and $F_{\frac{z-i}{10}}$, respectively, because $\frac{z\pm i}{10} > 14$. Now formulae (2.5)–(2.10) lead to

$$\alpha^{\frac{z-2}{2}} < \sqrt{F_z} < \alpha^{\left(\frac{z-i}{10}-1\right)+\left(\frac{z-i}{10}+0.5\right)+\left(\frac{z+i}{10}+0.5\right)+\left(\frac{z+i}{10}+0.5\right)}, \tag{2.11}$$

which implies that

$$\frac{z-2}{2} < \frac{2z}{5} + 0.5,$$

contradicting the fact that $z > 150$.

From now on, we analyze those cases when at least one of the numbers d_k for $k = 1,\ 2,\ 3,\ 4$, which we will denote by d, is less than five.

First assume that $d = 4$. Then either $\frac{z+\eta_1 i}{8} = \frac{y+\eta_2 j}{2}$, or $\frac{z+\eta_1 i}{8} = \frac{y+\eta_2 j}{6}$, where $\eta_1,\ \eta_2 \in \{\pm 1\}$.

If the first equality holds, then Lemma 2.7 leads to $z = 4y + 4\eta_2 j - \eta_1 i \leq 2y$. Thus, $z \leq 2y \leq \eta_1 i - 4\eta_2 j \leq 10$, contradicting the fact that $z > 150$.

The second equality leads to $y = \frac{3z+3\eta_1 i - 4\eta_2 j}{4}$. In this case,

$$\frac{y + \eta_2' j}{2} = \frac{3z + 3\eta_1 i + tj}{8}, \tag{2.12}$$

where $t = 4(\eta_2' - \eta_2) \in \{\pm 8, 0\}$ for $\eta_2' \in \{\pm 1\}$. Applying Lemma 2.5, we get

$$\begin{aligned} \gcd\left(\frac{z+\eta_1' i}{2}, \frac{y+\eta_2' j}{2}\right) &= \gcd\left(\frac{z+\eta_1' i}{2}, \frac{3z+3\eta_1 i + tj}{8}\right) \\ &\leq \left|\frac{3(\eta_1'-\eta_1)i - tj}{2}\right| \leq 14, \end{aligned} \tag{2.13}$$

for all $(\eta_1', \eta_2') \neq (\eta_1, \eta_2) \in \{\pm 1\}^2$. For the last inequality above, we used Lemma 2.5 together with the fact that $3(\eta_1' - \eta_1) - tj \neq 0$. Indeed, if $3(\eta_1 - \eta_1') - tj = 0$, then $3 \mid tj$, and since $t \in \{\pm 8, 0\}$, $j \in \{\pm 1, \pm 2\}$, we get that $t = 0$, therefore $\eta_2 = \eta_2'$. Since also $3(\eta_1 - \eta_1') = tj = 0$, we get $\eta_1 = \eta_1'$, therefore $(\eta_1', \eta_2') = (\eta_1, \eta_2)$, which is not allowed.

Continuing with the case $d = 4$, since $F_{14} < L_{14} = 843$ and $\frac{z \pm i}{8} > 18$, we get that $\varepsilon_2 < 0.25$ and $\delta_2 < 0.25$, where these values correspond to $L_{\frac{z\pm i}{8}}$ and $F_{\frac{z\pm i}{8}}$, respectively. It now follows that

$$\alpha^{\frac{z-2}{2}} < \alpha^{\frac{z\pm i}{8}+0.25} L_{14}^3 \leq 843^3 \alpha^{\frac{z+2}{8}+0.25}.$$

Thus, $z < 4 + 8\log_\alpha 843 < 116$, which completes the analysis for this case.

Consider now the case $d = 3$. The only possibility is $\frac{z+\eta_1 i}{6} = \frac{y+\eta_2 j}{2}$ for some $\eta_1, \eta_2 \in \{\pm 1\}$. Together with Lemma 2.7, we get $z = 3y + 3\eta_2 j - \eta_1 i \leq 2y$. Consequently, $\frac{z}{2} \leq y \leq \eta_1 i - 3\eta_2 j \leq 8$, which is impossible.

Assume next that $d = 2$. Then $\frac{z+\eta_1 i}{4} = \frac{y+\eta_2 j}{2}$ for some $\eta_1, \eta_2 \in \{\pm 1\}$. We get that $y = \frac{z+\eta_1 i - 2\eta_2 j}{2}$. Thus, $\frac{y+\eta_2' j}{2} = \frac{z+\eta_1 i + tj}{4}$ with $t = 2(\eta_2' - \eta_2) \in \{\pm 4, 0\}$. By Lemma 2.5, we have

$$\begin{aligned} \gcd\left(\frac{z+\eta_1' i}{2}, \frac{y+\eta_2' j}{2}\right) &= \gcd\left(\frac{z+\eta_1' i}{2}, \frac{z+\eta_1 i + tj}{4}\right) \\ &\leq |(\eta_1'-\eta_1)i - tj| \leq 12. \end{aligned}$$

The argument works assuming that the last number above is not zero for $(\eta_1', \eta_2') \neq (\eta_1, \eta_2) \in \{\pm 1\}^2$. Assume that it is. Then $(\eta_1' - \eta_1)i = tj$. Clearly, tj is always a multiple of 4. If it is zero, then $t = 0$, so $\eta_2' = \eta_2$. Then also $(\eta_1 - \eta_1')i = tj = 0$, therefore $\eta_1' = \eta_1$. Hence, $(\eta_1', \eta_2') = (\eta_1, \eta_2)$, which is not allowed. Assume now that $t \neq 0$. Then $(\eta_1 - \eta_1')i \neq 0$, so $\eta_1' = -\eta_1$. Also, $t \neq 0$, therefore $\eta_2 = -\eta_2'$. We get that $2\eta_1 i = -4\eta_2 j$, therefore $\eta_1 i = -2\eta_j$. Thus, $i = \pm 1$ and $j = \pm 2$. In particular, z is odd and y is even. Now $(z + \eta_1 i)/2$ is divisible by a larger power of 2 than $(y + \eta_2 j)/2$. A quick inspection of formulae (2.7)–(2.10) defining m_1, m_2, m_3 and m_4 together with Lemma 2.1 (ii) and (iii), shows that the only interesting cases are when $k = 1$ or 2 (since $m_3 \mid 2$ and $m_4 \mid 2$). Thus, $(\eta_1, \eta_2) = (-1, -1)$ or $(-1, 1)$. Hence, $(\eta_1', \eta_2') = (1, 1)$ or $(1, -1)$, and here we have that $m_3 \mid 2$ and $m_4 \mid 2$ anyway. This takes care of the case when $(\eta_1' - \eta_1)i - tj = 0$.

Continuing with $d=2$, since $\frac{z\pm i}{4}\geq 37$, Lemma 2.3 yields $\varepsilon_2,\delta_2<0.1$. We then get the estimate

$$\alpha^{\frac{z-2}{2}}<\alpha^{\frac{z\pm i}{4}+0.1}L_{12}^3\leq 322^3\alpha^{\frac{z+2}{4}+0.1},$$

leading to $z<6.4+12\log_\alpha 322<150.5$, which is a contradiction.

Finally, we assume that $d=1$. The equality $\frac{z\pm i}{2}=\frac{y\pm j}{2}$ leads to $z=y\mp i\pm j$. Obviously, here $\mp i\pm j$ must be positive, otherwise we would get $z\leq y$. Note that in the application of Lemma 2.2, both z and y are classified according to their congruence classes modulo 4. The following table summarizes the critical cases of $d=1$. Only 6 layouts in Table 1 below need further investigations (the sign † abbreviates a contradiction).

	$(z,y)\ \ (4)$	(i,j)	possible equalities	consequence
1	$(1,1)$	$(1,-1)$	$\frac{z-1}{2}=\frac{y+1}{2}$	$z=y+2$ † : $x\equiv y \pmod 4$
2	$(1,2)$	$(1,-2)$ $(-1,-2)$	$\frac{z-1}{2}=\frac{y+2}{2}$ $\frac{z+1}{2}=\frac{y+2}{2}$	$z=y+3$ † : d_2 must be even $z=y+1$ † : $z\equiv y-1 \pmod 4$
3	$(1,3)$	$(1,-1)$	$\frac{z-1}{2}=\frac{y+1}{2}$	$z=y+2$ † : d_1 must be even
4	$(1,0)$	$(1,-2)$ $(-1,-2)$	$\frac{z-1}{2}=\frac{y+2}{2}$ $\frac{z+1}{2}=\frac{y+2}{2}$	$z=y+3$ † : $x\equiv y+1 \pmod 4$ $z=y+1$ **is possible** $(d_3=1)$
5	$(3,1)$	$(1,-1)$	$\frac{z-1}{2}=\frac{y+1}{2}$	$z=y+2$ **is possible** $(d_4=1)$
6	$(3,2)$	$(-1,-2)$ $(1,-2)$	$\frac{z+1}{2}=\frac{y+2}{2}$ $\frac{z-1}{2}=\frac{y+2}{2}$	$z=y+1$ † : d_2 must be even $z=y+3$ † : $x\equiv y+1 \pmod 4$
7	$(3,3)$	$(1,-1)$	$\frac{z-1}{2}=\frac{y+1}{2}$	$z=y+2$ † : $x\equiv y \pmod 4$
8	$(3,0)$	$(-1,-2)$ $(1,-2)$	$\frac{z+1}{2}=\frac{y+2}{2}$ $\frac{z-1}{2}=\frac{y+2}{2}$	$z=y+1$ † : $x\equiv y-1 \pmod 4$ $z=y+3$ **is possible** $(d_3=1)$
9	$(2,1)$	$(2,1)$ $(2,-1)$	$\frac{z-2}{2}=\frac{y-1}{2}$ $\frac{z-2}{2}=\frac{y+1}{2}$	$z=y+1$ † : d_1 must be even $z=y+3$ † : $x\equiv y+1 \pmod 4$
10	$(2,2)$	$(2,-2)$	$\frac{z-2}{2}=\frac{y+2}{2}$	$z=y+4$ † : d_2 must be even
11	$(2,3)$	$(2,-1)$ $(2,1)$	$\frac{z-2}{2}=\frac{y+1}{2}$ $\frac{z-2}{2}=\frac{y-1}{2}$	$z=y+3$ † : d_1 must be even $z=y+1$ † : $x\equiv y-1 \pmod 4$
12	$(2,0)$	$(2,-2)$	$\frac{z-2}{2}=\frac{y+2}{2}$	$z=y+4$ † : $x\equiv y+2 \pmod 4$
13	$(0,1)$	$(2,1)$ $(2,-1)$	$\frac{z-2}{2}=\frac{y-1}{2}$ $\frac{z-2}{2}=\frac{y+1}{2}$	$z=y+1$ † : $x\equiv y-1 \pmod 4$ $z=y+3$ **is possible** $(d_4=1)$
14	$(0,2)$	$(2,-2)$	$\frac{z-2}{2}=\frac{y+2}{2}$	$z=y+4$ † : $x\equiv y+2 \pmod 4$
15	$(0,3)$	$(2,-1)$ $(2,1)$	$\frac{z-2}{2}=\frac{y+1}{2}$ $\frac{z-2}{2}=\frac{y-1}{2}$	$z=y+3$ † : $x\equiv y+1 \pmod 4$ $z=y+1$ **is possible** $(d_4=1)$
16	$(0,0)$	$(2,-2)$	$\frac{z-2}{2}=\frac{y+2}{2}$	$z=y+4$ **is possible** $(d_3=1)$

TABLE 1. The case $d=1$.

In what follows, we consider separately the 6 exceptional cases. The common treatment of all these cases is to go back to the system (1.3). In all exceptional cases we have $z = y + s$, where $s \in \{1, 2, 3, 4\}$. Hence,

$$\left\{ \begin{array}{rcl} ab + 1 & = & F_x, \\ ac + 1 & = & F_{z-s}, \\ bc + 1 & = & F_z, \end{array} \right. \tag{2.14}$$

and, as previously, $c \mid \gcd(F_{z-s} - 1, F_z - 1)$.

TABLE 1, ROW 4. $z \equiv 1,\ y \equiv 0 \pmod 4$, $z = y + 1$ and

$$F_{z-1} - 1 = F_{\frac{z+1}{2}} L_{\frac{z-3}{2}}, \qquad F_z - 1 = F_{\frac{z-1}{2}} L_{\frac{z+1}{2}}.$$

Clearly,

$$\gcd\left(F_{\frac{z+1}{2}}, F_{\frac{z-1}{2}}\right) = 1, \quad \gcd\left(L_{\frac{z-3}{2}}, L_{\frac{z+1}{2}}\right) = 1, \quad \gcd\left(F_{\frac{z+1}{2}}, L_{\frac{z+1}{2}}\right) = 1,\ 2,$$

while

$$\gcd(L_{\frac{z-3}{2}}, F_{\frac{z-1}{2}}) = \left\{ \begin{array}{l} L_{\gcd\left(\frac{z-3}{2}, \frac{z-1}{2}\right)} = L_1 = 1 \\ 1 \text{ or } 2 \end{array} \right\} \le 2.$$

Therefore $c \le 4$, and we arrived at a contradiction because $F_z = bc + 1 \le 13$ contradicts $z > 150$.

TABLE 1, ROW 5. $z \equiv 3,\ y \equiv 1 \pmod 4$, $z = y + 2$ and

$$F_{z-2} - 1 = F_{\frac{z-3}{2}} L_{\frac{z-1}{2}}, \qquad F_z - 1 = F_{\frac{z+1}{2}} L_{\frac{z-1}{2}}.$$

Since

$$\gcd\left(F_{\frac{z-3}{2}}, F_{\frac{z+1}{2}}\right) = 1,$$

we get $c \mid \gcd(F_{z-2} - 1, F_z - 1) = L_{\frac{z-1}{2}}$. Consequently, by the proof of Lemma 2.7,

$$L_{\frac{z-1}{2}} = c_1 c > c_1 \sqrt{F_z}.$$

By Lemma 2.6, we now have

$$c_1 < \frac{L_{\frac{z-1}{2}}}{\sqrt{F_z}} < 2.$$

Hence, $c_1 = 1$, therefore $c = L_{\frac{z-1}{2}}$. In view of equation (2.14), we get $a = F_{\frac{z-3}{2}}$, $b = F_{\frac{z+1}{2}}$, and so

$$F_x = F_{\frac{z-3}{2}} F_{\frac{z+1}{2}} + 1 = F^2_{\frac{z-1}{2}} + (-1)^{\frac{z-1}{2}} + 1 = F^2_{\frac{z-1}{2}}. \tag{2.15}$$

By the work of Cohn [4], we get that (2.15) is not possible for $z > 150$.

TABLE 1, ROW 8. $z \equiv 3,\ y \equiv 0 \pmod 4$, $z = y + 3$ and

$$F_{z-3} - 1 = F_{\frac{z-1}{2}} L_{\frac{z-5}{2}}, \qquad F_z - 1 = F_{\frac{z+1}{2}} L_{\frac{z-1}{2}}.$$

It follows easily, by Lemma 2.1, that

$$\gcd\left(F_{\frac{z-1}{2}}, F_{\frac{z+1}{2}}\right) = 1, \quad \gcd\left(L_{\frac{z-5}{2}}, L_{\frac{z-1}{2}}\right) = 1, \quad \gcd\left(F_{\frac{z-1}{2}}, L_{\frac{z-1}{2}}\right) = 1,\ 2,$$

and

$$\gcd\left(L_{\frac{z-5}{2}}, F_{\frac{z+1}{2}}\right) = \left\{ \begin{array}{l} L_{\gcd(\frac{z+1}{2},\frac{z-5}{2})} \le L_3 = 4 \\ 1 \text{ or } 2 \end{array} \right\} \le 4.$$

Thus, $c \mid \gcd(F_{z-3}-1, F_z-1) \le 8$. However, the inequalities $a < b < c \le 8$ contradict the fact that $z > 150$.

Table 1, Row 13. $z \equiv 0,\ y \equiv 1 \pmod 4$, $z = y+3$ and

$$F_{z-3} - 1 = F_{\frac{z-4}{2}} L_{\frac{z-2}{2}}, \qquad F_z - 1 = F_{\frac{z+2}{2}} L_{\frac{z-2}{2}}. \tag{2.16}$$

Since

$$\gcd\left(F_{\frac{z-4}{2}}, F_{\frac{z+2}{2}}\right) = F_{\gcd\left(\frac{z-4}{2},\frac{z+2}{2}\right)} \le F_3 = 2,$$

we have $c \mid \gcd(F_{z-2}-1, F_z-1) = L_{\frac{z-1}{2}}$, or $c \mid \gcd(F_{z-2}-1, F_z-1) = 2L_{\frac{z-1}{2}}$.

In the first case, we get

$$L_{\frac{z-2}{2}} = c_2 c > c_2\sqrt{F_z},$$

and applying Lemma 2.6 we arrive at

$$c_2 < \frac{L_{\frac{z-1}{2}}}{\sqrt{F_z}} < 1,$$

which is a contradiction.

In the second case, put

$$2L_{\frac{z-2}{2}} = c_3 c > c_3\sqrt{F_z}.$$

Again by Lemma 2.6, we obtain

$$c_3 < \frac{2L_{\frac{z-1}{2}}}{\sqrt{F_z}} < 2.$$

Thus, $c_3 = 1$, therefore $c = 2L_{\frac{z-1}{2}}$. System (2.14) and relations (2.16) lead to $2a = F_{\frac{z-4}{2}}$, $2b = F_{\frac{z+2}{2}}$, and

$$F_x = \frac{1}{4} F_{\frac{z-4}{2}} F_{\frac{z+2}{2}} + 1.$$

On the one hand, since $z > 150$, by Lemma 2.3, we get

$$\alpha^{x-1.67} > F_x > \frac{1}{4}\alpha^{\frac{z-4}{2}-1.68}\alpha^{\frac{z+2}{2}-1.68} > \alpha^{z-1-3.36-2.89},$$

therefore $x > z - 5.48$. On the other hand, by combining Lemma 2.3 and Lemma 2.4 with $\kappa < 0.01$, we get

$$\alpha^{x-1.68} < F_x < \frac{1}{4}\alpha^{\frac{z-4}{2}-1.67}\alpha^{\frac{z+2}{2}-1.67} + 1 < \alpha^{z-1-3.34-2.88+0.01},$$

leading to $x < z - 5.53$. But the interval $(z-5.48, z-5.53)$ does not contain any integer, which takes care of this case.

TABLE 1, ROW 15. $z \equiv 0,\ y \equiv 3 \pmod 4$, $z = y + 1$ and

$$F_{z-1} - 1 = F_{\frac{z}{2}} L_{\frac{z-2}{2}}, \qquad F_z - 1 = F_{\frac{z+2}{2}} L_{\frac{z-2}{2}}.$$

Since

$$\gcd(F_{\frac{z}{2}}, F_{\frac{z+2}{2}}) = 1,$$

we get $c \mid \gcd(F_{z-1}-1, F_z-1) = L_{\frac{z-2}{2}}$. Consequently, by the proof of Lemma 2.7, it follows that

$$L_{\frac{z-2}{2}} = c_4 c > c_4 \sqrt{F_z}.$$

Now Lemma 2.6 leads to the contradiction

$$c_4 < \frac{L_{\frac{z-2}{2}}}{\sqrt{F_z}} < 1.$$

TABLE 1, ROW 16. $z \equiv 0,\ y \equiv 0 \pmod 4$, $z = y + 4$ and

$$F_{z-4} - 1 = F_{\frac{z-2}{2}} L_{\frac{z-6}{2}}, \qquad F_z - 1 = F_{\frac{z+2}{2}} L_{\frac{z-2}{2}}.$$

Obviously,

$$\gcd(F_{\frac{z-2}{2}}, F_{\frac{z+2}{2}}) = 1, \qquad \gcd(L_{\frac{z-6}{2}}, L_{\frac{z-2}{2}}) = 1, \qquad \gcd(F_{\frac{z-2}{2}}, L_{\frac{z-2}{2}}) = 1,\ 2,$$

while

$$\gcd(L_{\frac{z-6}{2}}, F_{\frac{z+2}{2}}) = \left\{ \begin{array}{l} L_{\gcd(\frac{z-6}{2}, \frac{z+2}{2})} \le L_4 = 7 \\ 1 \text{ or } 2 \end{array} \right\} \le 7.$$

Thus, $c \le 14$, which leads to a contradiction with $z > 150$.

The proof of the Theorem 1.2 is now complete.

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