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Side-side-angle with fixed angles and the cyclic quadrilateral

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Abstract. Among the triangle congruence axioms, the side-side-angle (SsA) axiom states that two triangles are congruent if and only if two pairs of corresponding sides and the angles opposite the longer sides are equal. The modification of the SsA axiom provides a construction with two triangle sequences . We require that the opposite angles of the equivalent shorter sides be fixed and the longer sides be equal. The locus of the intersection points of other sides of triangles is derived to be a hyperbola, and in a generalized form defined by a complete quadrilateral, it is a conic section.

Mathematics Subject Classification. Primary 51N20, 51N05; Secondary 51N35.

Keywords. Triangle congruence, side-side-angle axiom, cyclic quadrilateral, conic section.

1. Introduction and main results

Triangle congruences play a significant role in Euclidean geometry, moreover, in absolute geometry. Some of them are axioms in almost all geometric structures. The triangle congruences are as follows, where S, A, s, R, and H represent, respectively, the side, angle, shorter side, right-angle, and hypotenuse.

- SSS Two triangles are congruent iff their corresponding sides are equal.
- ASA Two triangles are congruent iff two pairs of corresponding angles and the sides between them are equal.
- SAS Two triangles are congruent iff two pairs of corresponding sides and the angles between those sides are equal.
- AAS Two triangles are congruent iff two pairs of corresponding angles and a non-included side are equal.

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SsA Two triangles are congruent iff two pairs of corresponding sides and the angles opposite the longer sides are equal.

RHS Two right-angled triangles are congruent iff their corresponding hypotenuse and one side are equal.

Currently, the triangle congruence axioms have attracted the interest of a number of researchers, such as Donnelly [3–5] and Righy [9], who examined their role in absolute geometry, Hähl and Peters [7], who derived a variant of Hilbert's axioms from a subset of them.

The SsA triangle congruence is the most complex of the triangle congruences. Some researchers disagree that it is a fundamental triangle congruence axiom. We understand that two triangles are not necessarily congruent if two pairs of corresponding sides and the angles opposite the shorter sides are equal. In this case, two non-congruent triangles are possible under these conditions. This led us to examine the following connections between these two triangles and the condition sSA:

Condition sSA holds for two triangles if their two pairs of corresponding sides and the angles opposite the shorter sides are equal.

Csiba and Németh [2] fixed and superimposed the shorter sides of two triangles under the condition sSA, and they demonstrated that the locus of the intersection points of the side lines is a hyperbola. In this article, with similar conditions we show that the locus of the intersection points of the side lines is a hyperbola if the angles opposite the shorter sides are also fixed.

Theorem 1.1. *Let A and B be two fixed points. If the triangles ABC and ABC', with condition sSA have their corresponding shorter sides AB, and the common angles $\gamma = \gamma'$ opposite the shorter sides are fixed, then the locus of the intersection points of the corresponding lines of other sides is a orthogonal (normal) hyperbola.*

In our construction, the points A, B, C, and C' in any position form a cyclic quadrilateral $\square ABCC'$ and as C moves on their circumcircle the resulting lines CC' are parallel. It means that they have a common infinite point. In the second part of our article, we give a generalization of our construction when lines CC' have a given fixed (not only infinity) point.

Cyclic quadrilaterals have a wide literature [6,8], one of the recent article about complete quadrilateral and quadrangle is [10]. However, to the best of our knowledge the following theorems about cyclic quadrilaterals are not known.

Theorem 1.2. *Let A, B, C, and C' be four points of a circle $\mathcal{K}$, where line CC' lies on another fixed (finite or infinite) point F. Then the locus of two diagonal points (not on line CC') of cyclic quadrilateral $ABCC'$ is a conic if C is moving along $\mathcal{K}$.*

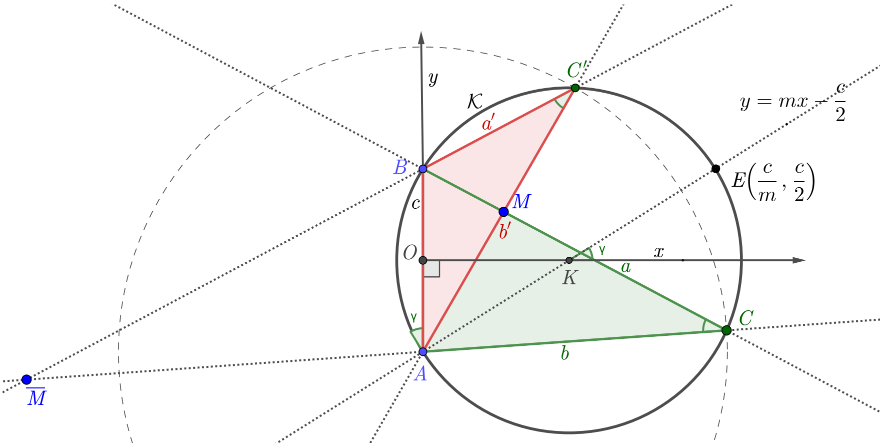

FIGURE 1 Base construction of sSA

In addition, the loci equations are derived in this article.

While writing this paper, we utilized GeoGebra software to visually verify our assumptions and to create the figures for the article, as well as Maple software to precisely validate some highly complex computations.

2. Base construction

2.1. Definitions

Consider two triangles ABC and $A'B'C'$ with the same orientation, sides, and angles a, b, c, α, β, γ, and a', b', c', α', β', γ', respectively. If $b = b'$, $c = c'$ with $b > c$ and $\gamma = \gamma'$, then both triangles satisfy the condition sSA. We are aware that these triangles are not necessarily congruent. Let $A = A'$, $B = B'$, and the third vertices of both triangles be in the same half-plane bordered by the line $c = c'$, since $b > c$. Figure 1 depicts our structure. Let us fix $\gamma = \gamma'$ $(0 \leq \gamma \leq 90°)$ and c. Let M and $\overline{M}$ be the intersection points of lines BC and $A'C'$, and lines AC and $B'C'$, respectively. In the following, we will look at this construction, specifically the locus of the points M and $\overline{M}$ when b is the variable. Because of continuity we allow the cases when C and C' are not the same sides of the line $c = c'$, but then $\gamma' = 180° - \gamma$, moreover, $b \leq c$ and $b' \leq c' = c$ (Fig. 2).

2.2. Locus of M

For our analytical calculations, we select an appropriate coordinate system. Let the origin O be the midpoint of segment AB and the y-axis be the line AB, so that the coordinates of A, B are $(0, -c/2)$ and $(0, c/2)$, as shown in Fig. 3. Moreover, let γ be fixed (in our figures $\gamma = 32°$). The points C

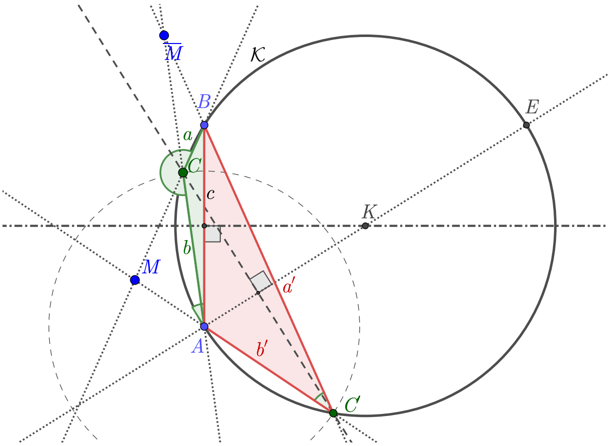

Figure 2 Extended base construction of sSA

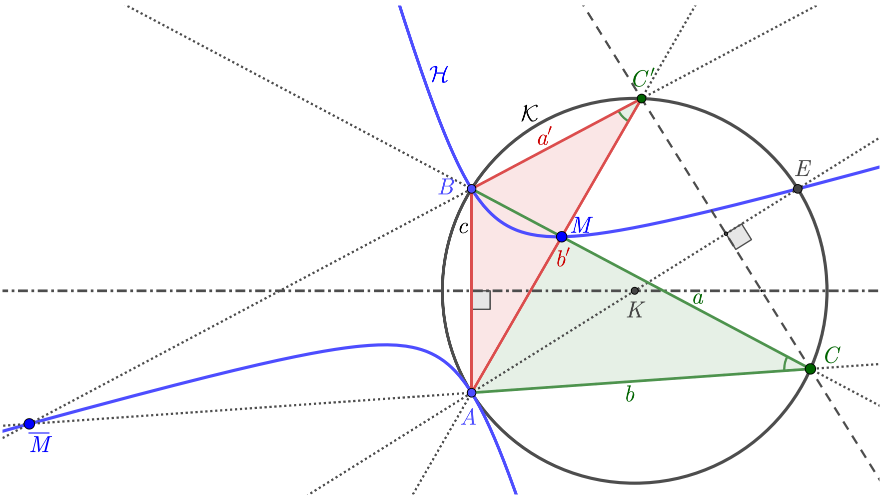

Figure 3 Hyperbola $\mathcal{H}$

and C' lie on the circumcircle $\mathcal{K}$ of $\square ABC'C$, whose equation, with center $K = (K_x, K_y) = (c/(2m), 0)$ and radius $\overline{AK}$, is

$$\left(x - \frac{c}{2m}\right)^2 + y^2 = \frac{c^2}{4} + \frac{c^2}{4m^2}, \tag{2.1}$$

where m is the slope of line AK and

$$m = \tan\gamma,$$

$m \neq 0$. The equation of line AK is $y = mx - c/2$, and is orthogonal to all of the lines which form an angle of γ with line AB.

The points $C = (C_x, C_y)$ and $C' = (C'_x, C'_y)$ are the same distance b from A. Thus, they lie on the circle $x^2 + (y + c/2)^2 = b^2$, where $0 \leq b \leq \sqrt{a^2 + c^2}$. From the intersection of this circle with $\mathcal{K}$, we get that their coordinates are

$$C = \left(\frac{(b + \Omega)mb}{c(m^2 + 1)}, \frac{b^2(1 + m^2) - (b + \Omega)^2}{2c(m^2 + 1)} \right),$$

$$C' = \left(\frac{(b - \Omega)mb}{c(m^2 + 1)}, \frac{b^2(1 + m^2) - (b - \Omega)^2}{2c(m^2 + 1)} \right),$$

where $\Omega = \sqrt{m^2(c^2 - b^2) + c^2}$.

The equations of lines BC and AC' are $2\Omega\, x + 2bm\, y - bcm = 0$ and $2(bm^2 + \Omega)/(\Omega - b)\, x + 2m\, y + cm = 0$, respectively. We get that their intersection point $M = (M_x, M_y)$ is

$$M = \left(\frac{bcm(b - \Omega)}{b^2m^2 + 2b\Omega - \Omega^2}, \frac{c(b^2m^2 + \Omega^2)}{2(b^2m^2 + 2b\Omega - \Omega^2)} \right).$$

If we change the role of C and C', then we obtain the point $\overline{M}$ with the same coordinates as M. Moreover, it means that the coordinates of $\overline{M}$ are obtained from the coordinates of M by replacing Ω with $-\Omega$. Thus, we have

$$\overline{M} = \left(\frac{bcm(b + \Omega)}{b^2m^2 - 2b\Omega - \Omega^2}, \frac{c(b^2m^2 + \Omega^2)}{2(b^2m^2 - 2b\Omega - \Omega^2)} \right).$$

Our variable is b and since the coordinates of points M and $\overline{M}$ are second order rational functions of b, then the loci of these points is a second order curve. Now, we shall show that this loci is the same hyperbola as we stated in Theorem 1.1.

Let E be the intersection point (different from A) of the line AK and the circle $\mathcal{K}$. Then its coordinates are $E = (c/m, c/2)$. Based on the previous construction and results, we formulate the following theorem.

Theorem 2.1. *The locus of the points M and $\overline{M}$ is the normal hyperbola $\mathcal{H}$ defined by the points A, B, E, and lines $2x + 2my \pm mc = 0$, as tangent lines at points A and B, respectively. The equation of $\mathcal{H}$ is*

$$-4x^2 + \frac{8xy}{m} + 4y^2 - c^2 = 0. \tag{2.2}$$

Proof. First, we know from the drawing construction that $b = b'$, and that the line AK is the symmetric axis of segment CC'. Thus, CC' is orthogonal to line AK with slope m (recall the equation of AK is $y = mx - c/2$). This implies that the slope of CC' is $-1/m$. We examine special cases as observations. If C is tends to B then $\overline{M}$ also tends to B, and consequently the line CC' becomes a tangent line t_B at B with equation $y = -1/mx + c/2$. We also see that when C tends to A, then M (and $\overline{M}$) tends to A, as well, and the line CC' tends to the tangent line t_A at A with equation $y = -1/mx - c/2$. The case when the length of $b = b'$ is maximal provides the point $E(c/m, c/2) = C = C' = M = \overline{M}$.

Using our above observations, let us define the hyperbola $\mathcal{H}$ with points A, B, E, and lines t_A and t_B as the tangents of $\mathcal{H}$ at points A and B, respectively.

The general equation of $\mathcal{H}$ is

$$h_1x^2 + h_2y^2 + h_3xy + h_4x + h_5y + h_6 = 0, \tag{2.3}$$

where the variables h_i are real numbers such that one of them is a free variable. If (x_0, y_0) is a point that serves as a pole with respect to $\mathcal{H}$, then its polar equation is

$$h_1x_0x + h_2y_0y + \frac{1}{2}h_3(x_0y + xy_0) + \frac{1}{2}h_4(x + x_0) + \frac{1}{2}h_5(y + y_0) + h_6 = 0. \tag{2.4}$$

If (x_0, y_0) is on $\mathcal{H}$, then its polar is a tangent of $\mathcal{H}$ at (x_0, y_0).

Thus, we have the system of linear equations for h_i from (2.3) and (2.4)

$$\begin{aligned}
c^2h_2 - 2ch_5 + 4h_6 &= 0,\\
c^2h_2 + 2ch_5 + 4h_6 &= 0,\\
4c^2h_1 + c^2m^2h_2 + 2c^2mh_3 + 4cmh_4 + 2mch_5 + 4m^2h_6 &= 0,\\
-2ch_2y - ch_3x + 2h_4x + h_5\,(2y - c) + 4h_6 &= \lambda_1(2x + 2my + cm),\\
2ch_2y + ch_3x + 2h_4x + h_5\,(2y + c) + 4h_6 &= \lambda_2(2x + 2my - cm).
\end{aligned}$$

We have two extra variables $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, which is important, as otherwise the last two equations would not be clearly defined as lines. Thus, the last two equations are polynomial equations and they yield linear equations for the equalities of the coefficients of x, y, and constant parts.

If we consider h_3 as the free variable, and let $h_3 = 2$, then by the solving of the system of linear equations we have $h_1 = -m, h_2 = m, h_4 = 0, h_5 = 0, h_6 = -(1/4)mc^2$. Thus the equation of $\mathcal{K}$ is

$$-4x^2 + \frac{8xy}{m} + 4y^2 - c^2 = 0.$$

Since the sum of the coefficients of x^2 and y^2 are zero and the coefficients of x and y are zeros, then $\mathcal{H}$ is a normal hyperbola.

We leave it to the reader to confirm that the coordinates of M satisfy Eq. 2.2. Moreover, if we change the positions of the points C and C', we shall obtain the point $\overline{M}$ instead of M. Consequently, $\overline{M}$ is one of the points of $\mathcal{H}$. □

From the proof of Theorem 2.1 we obtain the following corollary.

Corollary 2.1. *The tangent lines to $\mathcal{H}$ at the points A and B, and the lines CC' for any C are parallel. Moreover, they are perpendicular to the line AK.*

In the following, we shall give the canonical equation of $\mathcal{H}$. For this we will choose other coordinate axes through the origin. We denote these axes as the ξ-axis and η-axis, respectively. See Fig. 4.

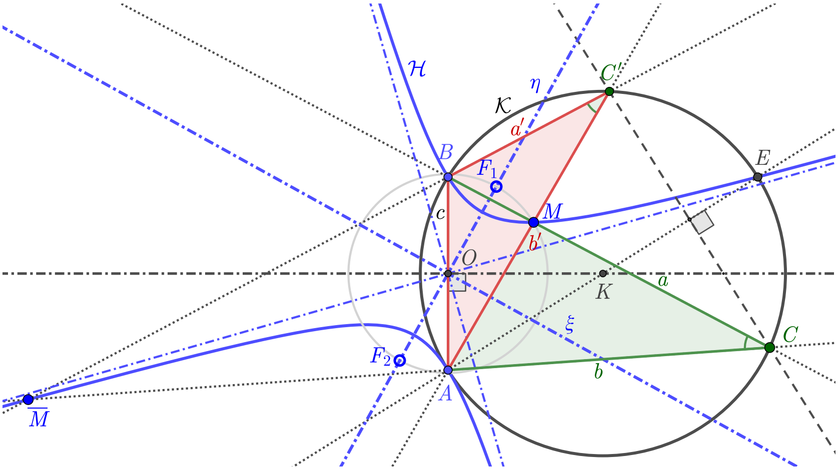

FIGURE 4 Axis of hyperbola $\mathcal{H}$

Theorem 2.2. *The canonical equation of $\mathcal{H}$ is*

$$-\frac{4\sqrt{m^2+1}}{mc^2}\,\xi^2+\frac{4\sqrt{m^2+1}}{mc^2}\,\eta^2=1, \tag{2.5}$$

where η and ξ are the transverse and conjugate axis of $\mathcal{H}$, respectively.

Proof. The matrix form of $\mathcal{H}$ given by (2.2) is

$$\vec{v}^{\,T}\mathbf{Q}\,\vec{v}=0, \tag{2.6}$$

where $\vec{v}=\begin{bmatrix}x\ y\ 1\end{bmatrix}$, and the symmetric coefficient matrix $\mathbf{Q}$ is

$$\mathbf{Q}=\begin{bmatrix}-4 & \frac{4}{m} & 0\\ \frac{4}{m} & 4 & 0\\ 0 & 0 & -c^2\end{bmatrix}.$$

Let $\mathbf{R}$ be a planar transformation given by a 3×3 matrix. In our case,

$$\mathbf{R}=\begin{bmatrix}\cos(\varrho) & -\sin(\varrho) & 0\\ \sin(\varrho) & \cos(\varrho) & 0\\ 0 & 0 & 1\end{bmatrix}$$

is a positive rotation with angle ϱ. Let ξ and η denote the new coordinate axes, so that

$$\vec{v}^{\,\prime}=\begin{bmatrix}\xi\ \eta\ 1\end{bmatrix}=\mathbf{R}\vec{v}.$$

Letting $\mathbf{R^{-1}}$ denote the inverse transformation of $\mathbf{R}$, then $\vec{v}=\mathbf{R^{-1}}\vec{v}^{\,\prime}$. Substituting into (2.6), we have that

$$\begin{aligned}\vec{v}^{\,T}\mathbf{Q}\,\vec{v}&=\left(\mathbf{R^{-1}}\vec{v}^{\,\prime}\right)^T\mathbf{Q}\,\left(\mathbf{R^{-1}}\vec{v}^{\,\prime}\right)\\&=(\vec{v}^{\,\prime})^T\left(\left(\mathbf{R^{-1}}\right)^T\mathbf{Q}\,\mathbf{R^{-1}}\right)\vec{v}^{\,\prime}=(\vec{v}^{\,\prime})^T\,\mathbf{Q}'\,\vec{v}^{\,\prime}=0.\end{aligned}$$

Since $\mathbf{R}$ is a rotation, then $\mathbf{R^{-1}} = \mathbf{R}^T$ and $\left(\mathbf{R^{-1}}\right)^T = \mathbf{R}$. Let

$$\varrho = \frac{\gamma}{2} - \frac{\pi}{4} \tag{2.7}$$

(the value of ϱ is a suggestion from the drawing of Fig. 4). Thus,

$$\mathbf{Q}' = \mathbf{R}\mathbf{Q}\mathbf{R}^T = \begin{bmatrix} -\dfrac{4}{\sin(\gamma)} & 0 & 0 \\ 0 & \dfrac{4}{\sin(\gamma)} & 0 \\ 0 & 0 & -c^2 \end{bmatrix}.$$

Thus, the rotation with angle (2.7) transforms the equation of $\mathcal{H}$ into the canonical form

$$-\frac{4}{c^2 \sin(\gamma)}\,\xi^2 + \frac{4}{c^2 \sin(\gamma)}\,\eta^2 = 1.$$

Recall, $m = \tan(\gamma)$. Another form of the canonical equation of $\mathcal{H}$ is (2.5). □

Moreover, in the (x, y)-coordinate system, the equations of the transverse and conjugate axis of $\mathcal{H}$ are $y = \cot(\varrho)x$ and $y = -1/\cot(\varrho)x$, respectively, and the coordinates of the foci in the (ξ, η)-system are $(0, \pm c\sqrt{\sin(\varrho)/2})$, where $\varrho = (\pi - 2\gamma)/4$.

3. Generalizations

According to Corollary 2.1, when C moves along the circle $\mathcal{K}$ the resulting lines CC' are parallel. Thus, they form a pencil of lines, and consequently have a common point at infinity. There is a very natural generalization of this problem in which the lines still form a pencil of lines, but have a common point which is not at infinity. More generally, we can consider a conic section instead of the circle $\mathcal{K}$. In this section, we give results for the more general case using projective tools. In the second subsection, we give the equation of the locus $\mathcal{H}$ of $\overline{M}$ with this classification.

3.1. Projective generalization

In this subsection, we generalize our construction for conics instead of the circle $\mathcal{K}$ in the sections above, and we prove the following theorem.

Theorem 3.1. *Let $\mathcal{K}$ be a non-degenerated conic and let A, B, C, and C' be its four points, where A, B are fix points, and the line CC' goes through another fix point F. We consider A, B, C, and C' as a complete quadrilateral with diagonal points M_1, M_2, and M_3 (see Fig. 5, M_3 is on the line AB). The locus of the diagonal points which are not on fix line AB is a conic when C is moving along $\mathcal{K}$.*

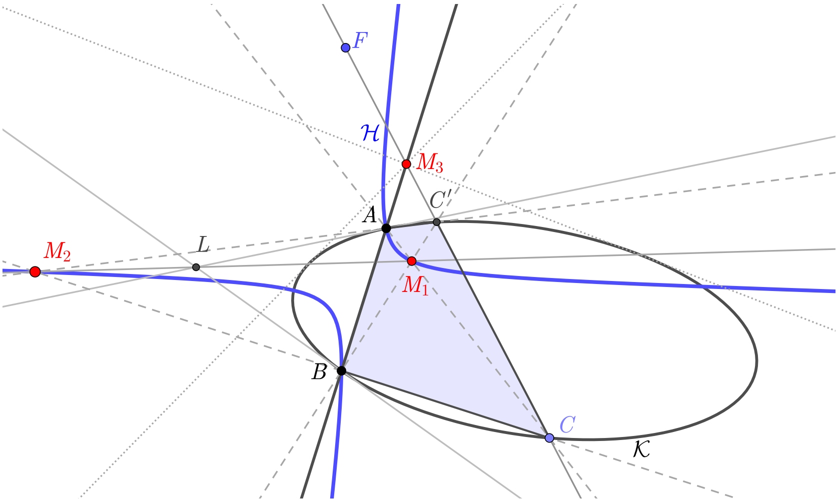

FIGURE 5 Projective generalization

Proof. We use projective geometry tools. We consider the lines passing through the point F as a pencil of lines and denote it by $[F]$. Let C_i and C_i' be the intersection points of one of the line of $[F]$ with $\mathcal{K}$ (see Fig. 6). If a line from $[F]$ is tangent to $\mathcal{K}$, then we denote the tangential point by E. If F is outside $\mathcal{K}$, then there are two such points, as illustrated in Fig. 6. Thus, $[F]$ generates a bijection among the points of $\mathcal{K}$ when C_i corresponds to C_i' and the tangential points correspond to themselves. This means that the cross ratios of any four points and their image points are the same. We now consider the pencils of lines $[A]$ and $[B]$, and define a map between them. We say that their lines correspond each other if they contain corresponding points of $\mathcal{K}$. Thus, AC_i and BC_i' are the corresponding lines with intersection point C_i. In this way, the cross ratios of any four lines and their image lines are the same. Thus, $[A]$ and $[B]$ are projective. According to the Steiner theory for projective pencils of lines, two pencils of lines at two different points are projective (perspective) if and only if the intersection points of corresponding lines lie on a non-degenerate (degenerate) conic. Thus, the points C_i form a conic $\mathcal{H}$. If $[A]$ and $[B]$ are perspective (ex., F is on AB or $\mathcal{K}$), then $\mathcal{H}$ is degenerated conic, otherwise not.

Moreover, from the properties of Steiner theory, the conic section $\mathcal{H}$ contains the points A and B as well.

If we change the roles of the points C_i and C_i', then we obtain similarly that the other diagonal point is on $\mathcal{H}$. □

Corollary 3.1. *If $\mathcal{H}$ is a non-degenerated conic, then its tangent lines at points A and B go through the point F.*

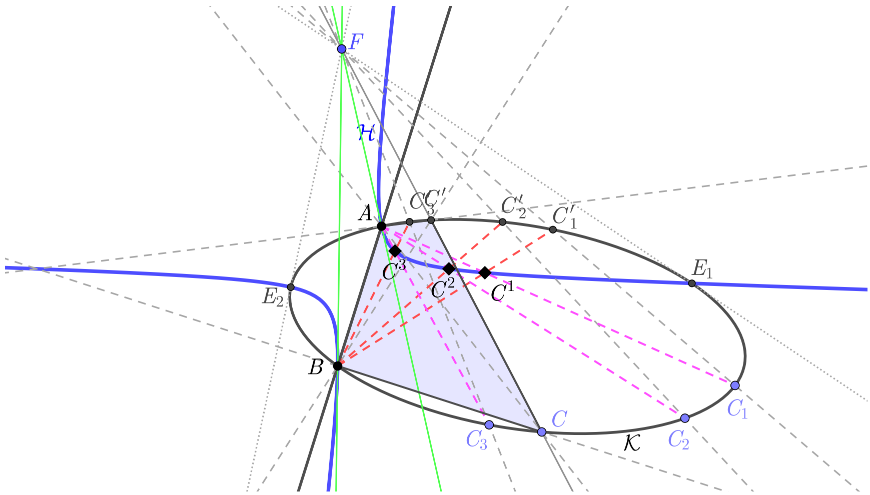

FIGURE 6 Intersection of projective pencils of lines

Proof. Let $C' = A$. We must prove that the line FAC has no other intersection points with $\mathcal{H}$. Suppose to the contrary that C_j is another intersection point. Consider the projective pencils of lines $[A]$ and $[B]$ (see the proof of Theorem 3.1 and Fig. 6). The line BC_j intersects the conic at C_j', and is different to A. This implies that F, C, C_j' and F, A, C are collinear, and that A, C, and C_j' are three different points of $\mathcal{H}$. Hence, we have a contradiction. □

Let L and H be the poles of the lines AB and CC' with respect to $\mathcal{H}$. The properties in the following lemma are well-known in projective geomety.

Lemma 3.2. *The polar line of M_3 with respect to $\mathcal{H}$ is the line M_1M_2. Moreover, the points L and H are also on this line.*

3.2. Generalization for the circle case

We now fix a circle $\mathcal{K}$ (see Figs. 7), and we follow the definitions from the previous sections. We have the following corollary of Theorem 3.1 which is equivalent to Theorem 1.2.

Corollary 3.3. *Let A and B be given points of the circle $\mathcal{K}$, and let F be an arbitrary point. Let the point C be a point of $\mathcal{K}$, and denote by C' the second intersection point of line FC with $\mathcal{K}$. Then the locus of the intersection points of AC with BC', and of AC' with BC is a conic section $\mathcal{H}$.*

In this section, we give the equation of $\mathcal{H}$ in a coordinate system similar to in the previous section, and we show how $\mathcal{H}$ depends on the position F in relation to the circle $\mathcal{K}$.

We choose a coordinate system similar to the one used in previous sections. The origin is the midpoint of AB and the y-axis is the line AB. However, we

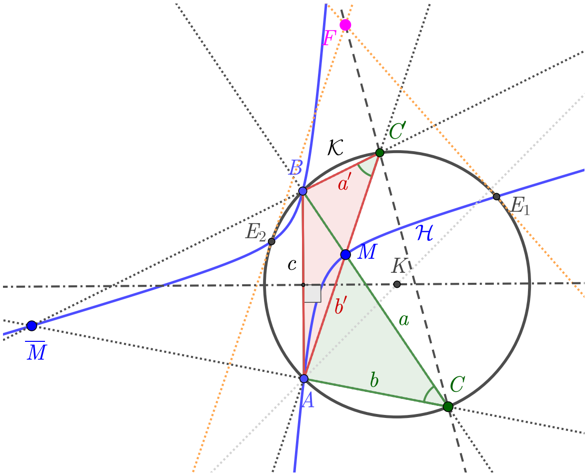

FIGURE 7 Generalization with circle and fix point

let the distant between the points A and B be $c = 2$, and we assume that A and B have coordinates $A = (0, -1)$ and $B = (0, 1)$. Moreover, let $\mathcal{K}$ be given with the equation

$$(x-h)^2 + y^2 = 1 + h^2, \tag{3.1}$$

where without the loss of generality, $0 \leq h$. In this coordinate system, let $F = (p, q)$, where $p \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{R}$. (If $p = 0$, then $\mathcal{H}$ is a degenerated conic, namely two intersecting lines, see later.)

To simplify our calculations we choose a point $T(0, t)$ on the y-axis, and we denote by C and C' the intersections of the line FT (with eq. $(t-q)x + py = pt$) with the circle $\mathcal{K}$.

Thus, we have

$$C = \left(\frac{(hp - qt + t^2 + \Theta)p}{p^2 + q^2 - 2qt + t^2}, \ \frac{hpq - hpt + p^2 t + (q-t)\,\Theta}{p^2 + q^2 - 2qt + t^2} \right)$$

and

$$C' = \left(\frac{hp - qt + t^2 - \Theta)p}{p^2 + q^2 - 2qt + t^2}, \ \frac{hpq - hpt + p^2 t - (q-t)\,\Theta}{p^2 + q^2 - 2qt + t^2} \right),$$

where $\Theta = \sqrt{h^2p^2 - 2hpqt + 2hpt^2 - p^2t^2 + p^2 + q^2 - 2qt + t^2}$.

The intersection point of the line AC' with line BC, and the intersection point of line AC with line BC', respectively, are

$$M = \left(\frac{-hp + qt - t^2 - t\Theta}{h^2p - 2hqt + 2ht^2 - pt^2}, \frac{-hq + ht - pt - h\Theta}{h^2p - 2hqt + 2ht^2 - pt^2} \right),$$

and

$$\overline{M} = \left(\frac{-hp + qt - t^2 + t\Theta}{h^2p - 2hqt + 2ht^2 - pt^2}, \frac{-hq + ht - pt + h\Theta}{h^2p - 2hqt + 2ht^2 - pt^2} \right).$$

The coordinates of points M and $\overline{M}$ are functions of t. Let us denote these functions by $M(t)$ and $\overline{M}(t)$, respectively. According to Corollary 3.3 we know that $\mathcal{H}$ is a conic, which implies that $\mathcal{H}$ and its equation are determined by its five points. Thus, the generalization of Eq. (2.3) is defined by the points $A = M(-1)$, $B = \overline{M}(1)$,

$$M(0) = \left(\frac{-1}{h}, \frac{-q - \sqrt{h^2p^2 + p^2 + q^2}}{hp} \right),$$

$$\overline{M}(0) = \left(\frac{-1}{h}, \frac{-q + \sqrt{h^2p^2 + p^2 + q^2}}{hp} \right),$$

and

$$M(q) = \left(\frac{-h - q\sqrt{h^2 - q^2 + 1}}{h^2 + q^2}, \frac{-q - h\sqrt{h^2 - q^2 + 1}}{h^2 + q^2} \right).$$

Substituting the coordinates of A and B into (2.3) we obtain the system of equations

$$\begin{aligned} h_2 - h_5 + h_6 &= 0, \\ h_2 + h_5 + h_6 &= 0, \end{aligned}$$

which implies that $h_5 = 0$ and $h_6 = -h_2$. Substituting the coordinates of $M(0)$, $\overline{M}(0)$ and $M(q)$ again into (2.3), then the system of equations is

$$\begin{aligned} h_1 M(0)_x^2 + h_2 M(0)_y^2 + h_3 M(0)_x M(0)_y + h_4 M(0)_x - h_2 &= 0, \\ h_1 \overline{M}(0)_x^2 + h_2 \overline{M}(0)_y^2 + h_3 \overline{M}(0)_x M(0)_y + h_4 \overline{M}(0)_x - h_2 &= 0, \\ h_1 M(q)_x^2 + h_2 M(q)_y^2 + h_3 M(q)_x M(q)_y + h_4 M(q)_x - h_2 &= 0. \end{aligned}$$

We consider h_2 to be a free variable, and we let $h_2 = -p$. From solving this system of equations we obtain $h_1 = p - 2h$, $h_3 = 2q$, $h_4 = -2$, $h_5 = 0$, $h_6 = p$.

Thus, the equation of $\mathcal{H}$ is

$$(p - 2h)\, x^2 + 2qxy - py^2 - 2x + p = 0 \tag{3.2}$$

with the coefficient matrix

$$\mathbf{Q}(\mathcal{H}) = \begin{bmatrix} p - 2h & q & -1 \\ q & -p & 0 \\ -1 & 0 & p \end{bmatrix}.$$

3.2.1. Classification of $\mathcal{H}$. For the classification of $\mathcal{H}$, we examine the determinant of the matrix $\mathbf{Q}(\mathcal{H})$ and some of its minor matrices. First of all, we suppose that $F \neq A$ and $F \neq B$.

It follows from

$$|\mathbf{Q}(\mathcal{H})| = -p\left(p^2 + q^2 - 2hp - 1\right),$$

that if $p = 0$ or $p^2 + q^2 - 2hp - 1 = 0$, then $|\mathbf{Q}(\mathcal{H})| = 0$ and $\mathcal{H}$ is degenerate. Otherwise, $\mathcal{H}$ is non-degenerate. If $p = 0$, then the point F is on the line AB.

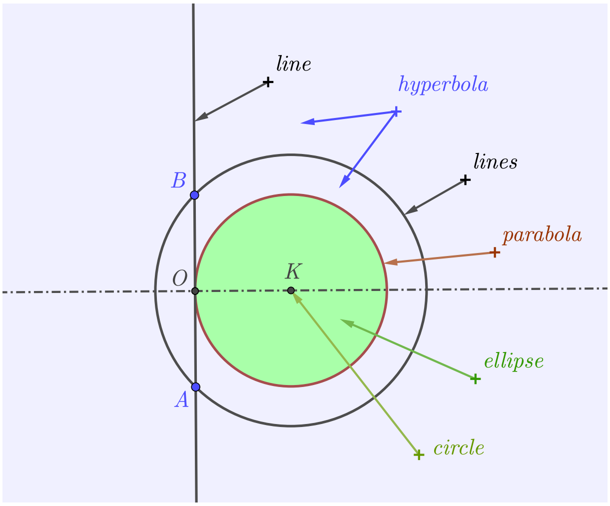

FIGURE 8 Classification of $\mathcal{H}$ depending on the position of F

If $p^2 + q^2 - 2hp - 1 = 0$, then F is a point of the circle $\mathcal{K}$ (substitute p and q into the Eq. (3.1)).

For the detailed classification, we have to give the minor

$M_{3,3}$ of matrix $\mathbf{Q}_{\mathcal{H}}$ (see [1]).

$$M_{3,3} = \begin{vmatrix} p-2h & q \\ q & -p \end{vmatrix} = -\left(p^2+q^2-2hp\right).$$

Let us note that $M_{3,3} = 0$ if and only if $p \neq 0$ and F is one of the points on the circle $\mathcal{C}$ with equation $(x-h)^2 + y^2 = h^2$ and concentric to $\mathcal{K}$ (Fig. 8).

Types of $\mathcal{H}$ ($F \neq A$ and $F \neq B$)

- If $|\mathbf{Q}(\mathcal{H})| = 0$, then $\mathcal{H}$ is degenerated, and
  - if $p = 0$, then (3.2) becomes $x(hx-qy+1) = 0$, where $hx-qy+1 = 0$ is the equation of **the polar of F with respect to $\mathcal{K}$** (see (2.4));
  - if $p^2 + q^2 - 2hp - 1 = 0$, then F is on the circle $\mathcal{K}$ and $M_{3,3} < 0$. Thus, $\mathcal{H}$ is **two intersecting lines** passing through F;
- If $|\mathbf{Q}(\mathcal{H})| \neq 0$, then $\mathcal{H}$ is non-degenerated, and
  - if $M_{3,3} < 0$, then F is outside of the circle $\mathcal{C}$ and $\mathcal{H}$ is **hyperbola**;
  - if $M_{3,3} = 0$, then F is on the circle $\mathcal{C}$ and $\mathcal{H}$ is **parabola**;
  - if $M_{3,3} > 0$, then F is inside of the circle $\mathcal{C}$ and $\mathcal{H}$ is **ellipse**.
    * If $F = K = (h, 0)$, so F is the center of $\mathcal{K}$, then $\mathcal{H}$ is a **circle** with equation $(x+h)^2 + y^2 = 1 + h^2$, see (3.2)), $\mathcal{H}$ is the mirror of $\mathcal{K}$ to the line AB.

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