Indag. Mathem., N.S., 13 (1), 131–142 March 25, 2002

The equations $2^N \pm 2^M \pm 2^L = z^2$*

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Communicated by Prof. R. Tijdeman at the meeting of November 26, 2001

1. INTRODUCTION

In the present paper we solve the title equations. It is easy to see that they lead either to

$$(1) \qquad 2^n \pm 2^m \pm 1 = x^2,$$

or to

$$(2) \qquad 2^n \pm 2^m \pm 2 = x^2.$$

While the examination of (2) is quite simple, as well as the resolution of $2^n \pm 2^m - 1 = x^2$, the equation

$$(3) \qquad 2^n + 2^m + 1 = x^2$$

requires more calculations and the application of some deep results of Beukers [2]. This problem has been posed by professor Tijdeman, and I heard it from Tengely.

From a wider point of view, equations of types similar to (1) and (2) have already been investigated. Gerono [4] proved that a Mersenne-number $M_k = 2^k - 1$ cannot be a power of a natural number if $k > 1$, so the equation

* Research supported by Hungarian National Foundation for Scientific Research Grant No. 25157/1998.

$2^k - 1 = x^2$ has only the solutions $(k, x) = (0, 0), (1, 1)$. For another example, it can readily be verified that $2^k + 1 = x^2$ implies $(k, x) = (3, 2)$.

Ramanujan [7] conjectured that the diophantine equation

$$(4) \qquad 2^k - 7 = x^2$$

has five solutions, namely $(k, x) = (3, 1)$, $(4, 3)$, $(5, 5)$, $(7, 11)$ and $(15, 181)$. His conjecture was first proved by Nagell [6]. The generalized Ramanujan-Nagell equation

$$(5) \qquad 2^k + D = x^2$$

in natural numbers k and x, where $D \neq 0$ is an integer parameter, was considered by several authors. See, for example, Apéry [1], Hasse [5], Beukers [2]. Taking $D = \pm\, 2^M \pm 2^L$, we investigate infinitely many generalized Ramanujan-Nagell equations.

Our main result is Theorem 1. Theorems 1 and 2 have interesting consequences connected to binary recurrences (Corollary 1). Finally, a corollary of Lemma 5 states that the ratio of two distinct triangular numbers cannot be a power of 4 (Corollary 2).

Acknowledgements. The author would like to thank professor Attila Pethő for the useful discussion we had on this subject matter. Further, thanks are also due to professors Robert Tijdeman and Yann Bugeaud for their kind help, and to the referee for the valuable remarks and suggestions.

2. RESULTS

Theorem 1. *If the positive integers n, m and x with $n \geq m$ satisfy*

$$(6) \qquad 2^n + 2^m + 1 = x^2,$$

then

(i) $(n, m, x) \in \{(2t, t+1, 2^t + 1) \mid t \in \mathbb{N},\ t \geq 1\}$ *or*
(ii) $(n, m, x) \in \{(5, 4, 7)\ ,\ (9, 4, 23)\}$.

Remarks. I. Equation (6), essentially, asks for odd natural numbers x whose squares contain exactly three 1 digits with respect to the base 2. Theorem 1 says that beside the infinite set $x^2 = 101_2^2 = 11001_2$, $1001_2^2 = 1010001_2$, $\ldots$, only $x^2 = 111_2^2 = 110001_2$ and $x^2 = 10111_2^2 = 1000010001_2$ possess the property above.

II. The solutions (i) and (ii) in Theorem 1 enable to determine all $n, m \in \mathbb{Z}$, $x \in Q$ satisfying (6).

Theorem 2. *If the positive integers n, m and x satisfy*

$$(7) \qquad 2^n - 2^m + 1 = x^2,$$

then

(i) $(n, m, x) \in \{(2t, t+1, 2^t - 1) \mid t \in \mathbb{N},\ t \geq 2\}$ or

(ii) $(n, m, x) \in \{(t, t, 1) \mid t \in \mathbb{N},\ t \geq 1\}$ or
(iii) $(n, m, x) \in \{(5, 3, 5),\ (7, 3, 11),\ (15, 3, 181)\}$.

Theorem 3. *If the positive integers n, m and x with $n \geq m$ satisfy*

$$2^n + 2^m - 1 = x^2, \tag{8}$$

then
(i) $(n, m, x) = (3, 1, 3)$.
Moreover, all the solutions of the equation

$$2^n - 2^m - 1 = x^2 \tag{9}$$

in positive integers n, m and x are given by
(ii) $(n, m, x) = (2,1,1)$.

One can find lots of results concerning occurrence of squares and higher powers in binary (or higher order) recurrences. See, for instance, Shorey, Tijdeman [8], Chapter 9. Corollary 1 determines all square terms in certain binary recursive sequences.

Corollary 1. *(Corollary of Theorems 1 and 2.) Let d be an arbitrarily fixed natural number. Consider the binary recurrences*

$$G_m = 3G_{m-1} - 2G_{m-2} \quad (m \geq 2), \qquad G_0 = 2^d + 2,\ G_1 = 2^{d+1} + 3; \tag{10}$$

$$H_m = 3H_{m-1} - 2H_{m-2} \quad (m \geq 2), \qquad H_0 = 2^d,\ H_1 = 2^{d+1} - 1. \tag{11}$$

(i) The only square occurring in the recursive sequence G is G_{d+2}, except for the following two cases. If $d = 1$, then G contains three squares, namely G_0, $G_3 = G_{d+2}$ and G_4. If $d = 5$, then G_4 and $G_7 = G_{d+2}$ are the squares in G.
(ii) If d is odd, then

$$H_m = w^2 \tag{12}$$

implies $m = d + 2$. If $d > 0$ is even, then equation (12) has exactly two solutions given by $m = 0$ and $m = d + 2$, except for three cases $d = 2, 4, 12$ when there is an additional square, viz. H_3.

The second corollary contributes to the colorful palette of the results concerning triangular numbers.

Corollary 2. *(Corollary of Lemma 5.) Let $\triangle_k$ denote the k^{th} triangular number, i.e. $\triangle_k = \frac{k(k+1)}{2}$, $k \geq 1$, $k \in \mathbb{N}$. Then the diophantine equation*

$$\frac{\triangle_y}{\triangle_x} = 4^t, \quad y \neq x \tag{13}$$

has no solution in natural numbers x, y and t.

Lemma 1. *Let $D_1 \in \mathbb{Z}$, $D_1 \neq 0$. If $|D_1| < 2^{96}$ and $2^n + D_1 = x^2$ has a solution (n, x) then*

$$n < 18 + 2\log_2 |D_1|. \tag{14}$$

Proof. This is Corollary 2 in [2] due to Beukers.

Lemma 2. *Let p be an odd power of 2. Then for all $x \in \mathbb{Z}$*

$$\left| \frac{x}{p^{0.5}} - 1 \right| > \frac{2^{-43.5}}{p^{0.9}}. \tag{15}$$

Proof. We refer again to Beukers, [2].

Lemma 3. *Let $D_2 \in \mathbb{N}$ be odd. The equation $2^n - D_2 = x^2$ has two or more solutions in positive integers n, x if and only if $D_2 = 7, 23$ or $2^k - 1$ for some $k \geq 4$. The solutions, in these exceptional cases, are given by the following table.*

$D_2 = 7$	$(n,x) = (3,1), (4,3), (5,5), (7,11), (15,181),$
$D_2 = 23$	$(n,x) = (5,3), (11,45),$
$D_2 = 2^k - 1, (k \geq 4)$	$(n,x) = (k,1), (2k-2, 2^{k-1} - 1).$

Proof. See Theorem 2 in [2].

Lemma 4. *All natural solutions (n, x) of the inequalities*

$$0 < |2^n - x^2| < 4 \tag{16}$$

in positive integers n and x are given by $(n, x) \in \{(1,1), (2,1), (3,3), (1,2)\}$.

Proof. In virtue of Lemma 1, $n < 22$ and the verification of all possible values n gives the solutions above. □

Lemma 5. *Let t be an arbitrary positive integer. If x and y are integers satisfying*

$$y^2 - 1 = 2^{2t}(x^2 - 1), \quad y > 1\ ,\ x > 1, \tag{17}$$

then $x = 2^{t-1}$ and $y = 2^{2t-1} - 1$ for $t > 1$.

Proof. It is easy to see that (17) is not solvable if $t = 1$. Suppose that $t > 1$, $y > 1$ and $x > 1$ satisfy (17). Then y is odd and

$$\frac{y-1}{2} \cdot \frac{y+1}{2} = 2^{2t-2}(x^2 - 1). \tag{18}$$

The greatest common divisor of $\frac{y-1}{2}$ and $\frac{y+1}{2}$ is 1 and $2^{2t-2} (\geq 4)$ divides exactly one of the terms on the left hand side of (18). Consequently, $y = 2^{2t-1}k \pm 1$

with some integer $k \geq 1$. By (17) we have $y < 2^t x$, therefore $2^{t-1}k \leq x$. Moreover, it follows that

$$(19) \qquad 2^{2t-2}k^2 - k = x^2 - 1 \qquad \text{or} \qquad 2^{2t-2}k^2 + k = x^2 - 1.$$

In the first case, clearly, $k = 1$ provides the solution $x = 2^{t-1}$, $y = 2^{2t-1} - 1$. If $k > 1$, then the inequalities

$$(20) \qquad x^2 = 2^{2t-2}k^2 - (k-1) < 2^{2t-2}k^2 \leq x^2$$

lead to contradiction.

In the second case of (19) it follows that

$$(21) \qquad (2^{t-1}k)^2 < (2^{t-1}k)^2 + k + 1 = x^2 < (2^{t-1}k + 1)^2,$$

which is impossible. □

Lemma 6. *Let n, m and x be positive integers satisfying $2 \leq m < n$ and*

$$(22) \qquad 2^n + 2^m + 1 = x^2.$$

Then $x = 2^{m-1}(2k+1) \pm 1$ with some $k \in \mathbb{N}$.

Proof. Assume that (n, m, x) is a solution of (22) under the assumptions made. For $m = 2$ the lemma trivially states that x is odd. If $m \geq 3$, then the congruence

$$(23) \qquad x^2 \equiv 1 \pmod{2^m}$$

has exactly four incongruent solutions, namely $x \equiv 1$, $x \equiv 2^{m-1} - 1$, $x \equiv 2^{m-1} + 1$ and $x \equiv 2^m - 1 \pmod{2^m}$.

The first and fourth cases are impossible because, by (22), $x = 2^m l \pm 1$, ($l \in \mathbb{N}$, $l \geq 1$) leads to

$$(24) \qquad 2^{n-m} + 1 = 2^m l^2 \pm 2l.$$

The second and third solutions of (23) provide

$$(25) \qquad x = 2^{m-1}(2k+1) \pm 1, \quad (k \in \mathbb{N}). \quad □$$

Lemma 7. *If n, m and x are natural numbers for which $m < n$ and $n < 2m - 2$, then*

$$(26) \qquad 2^n + 2^m + 1 = x^2$$

implies $(n, m, x) = (5, 4, 7)$.

Proof. The conditions of the lemma give $m \geq 4$. Suppose that (n, m, x) satisfy (26). Combining Lemma 6 and (26) we obtain

$$(27) \qquad 2^n + 2^m + 1 = r^2 2^{2m-2} \pm r2^m + 1,$$

where r is a positive odd integer. Since $2m - 2 \geq n + 1$, we get

$$(28)\qquad 2^m(1 \mp r) \geq (2r^2 - 1)2^n.$$

Hence $r = 1$, $x = 2^{m-1} - 1$ and $n \leq m + 1$. By $m < n$ we have $n = m + 1$ and we can conclude that $m = 4$, $n = 5$ and $x = 7$.

Lemma 8. *If D, k and x are positive integers, $k \geq 3$ and*

$$(29)\qquad 2^{D+8k} + 2^{4k} + 1 = x^2,$$

then $D > 56k - 32$.

Proof. Let $\nu_2(n)$ denote the 2-adic value of the integer n. Assume that the integers D, k and x satisfy the conditions of the lemma. By Lemma 6 we have two possibilities for x.

A) First consider the case $x = 2^{4k-1}(2u_0 + 1) + 1$, $(u_0 \geq 0)$. By (29) we obtain

$$(30)\qquad 2^{D+4k-1} = 2^{4k-3}(2u_0 + 1)^2 + u_0,$$

where u_0 must be positive and $u_0 = 2^{4k-3}u_1$ with some positive odd integer u_1. Otherwise, dividing (30) by $2^{\min\{4k-3, \nu_2(u_0)\}}$, it leads to contradition. In the sequel, this type of argument will be applied without any further notice. It follows that

$$(31)\qquad 2^{D+2} = 2^{8k-4}u_1^2 + 2^{4k-1}u_1 + (u_1 + 1).$$

Then $u_1 + 1 = 2^{4k-1}u_2$ for some suitable positive odd integer u_2, and by (31) we get

$$(32)\qquad 2^{D-4k+3} = 2^{4k-3}\left(2^{4k-1}u_2 - 1\right)^2 + 2^{4k-1}u_2 + (u_2 - 1).$$

Clearly, $u_2 \neq 1$, $u_2 - 1 = 2^{4k-3}u_3$, $(u_3 \in \mathbb{N}, u_3 \equiv 1 \pmod 2)$, further

$$(33)\qquad \begin{aligned} 2^{D-8k+6} &= 2^{8k-2}\left(2^{4k-3}u_3 + 1\right)^2 - 2^{4k}\left(2^{4k-3}u_3 + 1\right) \\ &\quad + 2^{4k-1}u_3 + (u_3 + 5). \end{aligned}$$

It is easy to see that $u_3 + 5 = 2^{4k-1}u_4$, where u_4 is an odd natural number. Hence

$$(34)\qquad \begin{aligned} 2^{D-12k+7} &= 2^{4k-1}\left(2^{4k-3}\left(2^{4k-1}u_4 - 5\right) + 1\right)^2 - \\ &\quad - 2^{4k-2}\left(2^{4k-1}u_4 - 5\right) + 2^{4k-1}u_4 + (u_4 - 7). \end{aligned}$$

By (34) we conclude that $u_4 - 7 = 2^{4k-2}u_5$. Here the odd integer $u_5 = \frac{u_4 - 7}{2^{4k-2}}$ is positive because $k \geq 3$ and $u_4 > 0$. It follows that

$$(35)\qquad \begin{aligned} 2^{D-16k+9} &= 2^{8k-5}\left(2^{4k-1}\left(2^{4k-2}u_5 + 7\right) - 5\right)^2 + 2^{8k-2}\left(2^{4k-2}u_5 + 7\right) - \\ &\quad - 2^{8k-3}u_5 + (u_5 - 12)2^{4k-1} + (u_5 + 21). \end{aligned}$$

Finally, $u_5 + 21 = 2^{4k-1}u_6$, $(u_6 \in \mathbb{N}, u_6 \equiv 1 \pmod 2)$, and then $u_6 - 33 = 2^{4k-4}u_7$ leads to the equality

(36) $$2^{D-24k+14} = \left(2^{4k-1}\left(2^{4k-2}Q_1 + 7\right) - 5\right)^2 + R_1 + S_1,$$

where

(37) $$Q_1 = 2^{4k-1}\left(2^{4k-4}u_7 + 33\right) - 21, \quad R_1 = 2^3\left(2^{4k-2}Q_1 + 7\right) - 2^2 Q_1,$$

(38) $$S_1 = 2^3\left(2^{4k-4}u_7 + 33\right) + u_7, \quad u_7 \in \mathbb{N},\ u_7 \equiv 1 \pmod 2.$$

Obviously, $Q_1 > 2^{8k-5}$, $S_1 > 0$, $R_1 > 0$. Therefore $2^{D-24k+14} > 2^{32k-16}$ and

(39) $$D > 56k - 30.$$

B) In the second case replace x by $2^{4k-1}(2u_0+1) - 1$, $(u_0 \geq 0)$ in (29) and, similarly as above, the substitutions $u_0 = 2^{4k-3}u_1 - 1$, $u_1 = 2^{4k-1}u_2 + 1$, $u_2 = 2^{4k-3}u_3 - 1$, $u_3 = 2^{4k-1}u_4 + 5$, $u_4 = 2^{4k-2}u_5 - 7$, $u_5 = 2^{4k-1}u_6 + 21$ and $u_6 = 2^{4k-4}u_7 - 33$ lead to the equality

(40) $$2^{D-24k+14} = \left(2^{4k-1}Q_2 + 5\right)^2 - 8Q_2 + R_2,$$

where

(41) $$Q_2 = 2^{4k-2}\left(2^{4k-1}\left(2^{4k-4}u_7 - 33\right) + 21\right) - 7,$$

(42) $$R_1 = 2^2\left(2^{4k-1}\left(2^{4k-4}u_7 - 33\right) + 21\right) - 2^3\left(2^{4k-4}u_7 - 33\right) - u_7,$$

and $u_7 \in \mathbb{N}$, $u_7 \equiv 1 \pmod 2$. It can be proved that $Q_2 > 2^{12k-8}$, $R_2 > 0$ and we have $2^{D-24k+14} > 2^{32k-18}$, which, together with (39), implies $D > 56k - 32$. The proof of Lemma 8 is complete. □

Lemma 9. *If a and c are non-negative integers satisfying*

(43) $$a^2 + (a+1)^2 = c^2,$$

then $a = 2P_nP_{n+1}$, $a + 1 = P_{n+1}^2 - P_n^2$ *or conversely, where* P_k *denotes the* k^{th} *term of the Pell sequence defined by* $P_0 = 0$, $P_1 = 1$ *and* $P_k = 2P_{k-1} + P_{k-2}$, $(k \geq 2)$.

Proof. Probably this is an old result. For the proof see, for instance, Cohn [3].

Proof of Theorem 1. Obviously, each element of the set

(44) $$T = \left\{(n,m) \in \mathbb{N}^2 \mid n = 2t, m = t+1, t \in \mathbb{N}, t \geq 1\right\}$$

(with some suitable $x \in \mathbb{N}$) satisfies the relations

(45) $$2^n + 2^m + 1 = x^2, \quad n \geq m \geq 1.$$

Let S denote the set of solutions (n,m) of (45), further let $M_1 = (5,4)$ and $M_2 = (9,4)$. We have to show that the set of exceptional solutions is $S \setminus T = \{M_1, M_2\}$.

Observe that $2^{n-m} + 1 = \frac{x^2-1}{2^m} \in \mathbb{N}$ if $(n,m) \in S$, further

(46) $$2^{2n-2m} + 2^{n-m+1} + 1 = \left(\frac{x^2-1}{2^m}\right)^2.$$

Hence a solution (n, m) of (45) provides $(2n - 2m, n - m + 1) \in S$, except when $2n - 2m < n - m + 1$, i.e. $n = m$. But Lemma 4 implies that the only solution (n, m) with $n = m$ is $(2, 2) \in T$.

In the sequel, we assume that $n > m$. Then the transformation

(47) $$\tau : (n, m) \longmapsto (2n - 2m, n - m + 1), \quad (n > m)$$

induces a map of $S \setminus \{(2, 2)\}$ into S.

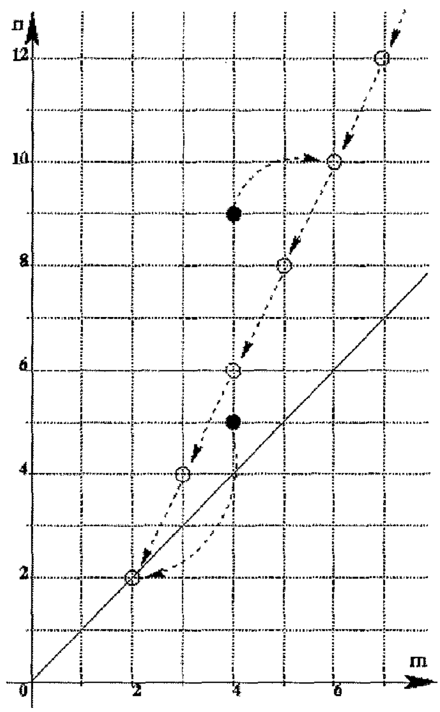

Figure 1. Map τ on the solutions of the equation $2^n + 2^m + 1 = x^2$.

The map τ has important properties. If $(n, m) \in S$, then let $\delta(n, m)$ denote the distance $n - m$ of the exponents n and m.

Property 1. $\delta(\tau(n, m)) = \delta(n, m) - 1$. In particular, $\tau(n, m) \neq (n, m)$, i.e. the map has no fixed points.

Property 2. If $(n, m) \in T \setminus \{(2, 2)\}$, more precisely if $(n, m) = (2t, t + 1)$, $t \geq 2$, then $\tau(n, m) = (2(t - 1), t) \in T$ is the '*lower neighbour*' solution of (n, m) in T. Thus the elements of the set T are ordered by τ. Moreover $\delta(\tau(2t, t + 1)) = t - 2$, $(t \geq 2)$ shows that all natural numbers occur as a difference of the exponents in the solution of (45).

Property 3. If (n,m) is an exceptional solution (i.e. $(n,m) \in S \setminus T$), then $\tau(n,m) \in T$ since $\tau(n,m) = (2\delta(n,m), \delta(n,m)+1)$. Especially, $\tau(5,4) = (2,2)$, $\tau(9,4) = (10,6)$.

If $m = 1$, then Lemma 4 implies a solution with $n < m$, which contradicts the assertion $n > m$.

Now suppose that the integers n and m satisfy $2 \leq m < n$ and (45). Reconsidering the map

$$\tau \,:\, S \setminus \{(2,2)\} \longmapsto S \tag{48}$$

with (47), by Properties 1-3 we have to prove that there are exactly two cases when $(n,m) \neq (n_1, m_1)$ and $\tau(n,m) = \tau(n_1, m_1)$. In other words, we must show that the system of the equations

$$2^n + 2^m + 1 = x^2 \tag{49}$$

$$2^{n+d} + 2^{m+d} + 1 = y^2 \tag{50}$$

in positive integers n, m, d, x, y with $2 \leq m < n$ has exactly two solutions.

Taking such a solution, obviously both $x > 1$ and $y > 1$ are odd. It follows from (49) and (50) that

$$y^2 - 1 = 2^d(x^2 - 1), \tag{51}$$

and by Lemma 5 we infer that d must be odd.

Observe that one of (n,m) and $(n+d, m+d)$ has to belong to the set $T \setminus \{(2,2)\}$. On the contrary, if both (n,m) and $(n+d, m+d)$ are exceptional, by the properties of the transformation τ there exists a solution $(n_2, m_2) \in T \setminus \{(2,2)\}$ such that $\tau(n_2, m_2) = \tau(n,m) = \tau(n+d, m+d)$. But in this case one of the distances $|n_2 - n| = |m_2 - m|$ and $|n_2 - (n+d)| = |m_2 - (m+d)|$ has to be even since d is odd, which contradicts again to Lemma 5. Therefore, we distinguish two cases.

A) First let (50) be the exceptional case, consequently $(n,m) \in T \setminus \{(2,2)\}$, and by (44) it follows that $n = 2m - 2$, which, together with (50), implies

$$2^{2m-2+d} + 2^{m+d} + 1 = y^2. \tag{52}$$

Here, if $m \geq 3$, then the exponents $m + d$ and $2m - 2 + d$ on the left hand side satisfy the conditions of Lemma 7. Thus we conclude that $m = 3$, $d = 1$, $y = 7$ is the only solution of (52) (and $n = 4$, $x = 5$ of (49)). It gives $M_1 \in S$. On the other hand, if $m = 2$, then $n = 2m - 2 \leq m$ leads to contradiction.

B) The second possibility is that (49) is the exceptional case, while $(n+d, m+d) \in T \setminus \{(2,2)\}$, i.e. $n = 2m + d - 2$. Then by (49) we have

$$2^{2m+d-2} + 2^m + 1 = x^2. \tag{53}$$

It is easy to show that one of the exponents must be even in (53). Since d is odd, therefore m has to be even. Put $m = 2r$, where $r \in \mathbb{N}$, $r \geq 1$, and let $D = d - 2$. If $D = -1$, then (53) is equivalent to

$$(54) \qquad 2^{4r-1} + 2^{2r} + 1 = x^2.$$

Observe that the left hand side of (54) is a sum of $(2^{2r-1})^2$ and $(2^{2r-1}+1)^2$, hence $a = 2^{2r-1}$, $a+1$ and x form a Pythagorean triple. Since a is even, by Lemma 9 we have $2^{2r-1} = 2P_nP_{n+1}$ with some $n \in \mathbb{N}$. Therefore, both P_n and P_{n+1} are power of 2, which is impossible if $n \geq 2$ because P_n and P_{n+1} are coprime. Since $P_0 = 0$, the only possibility is $n = 1$, but $1 \cdot 2 \neq 2^{2r-2}$.

Consequently, $D \geq 1$ and we have

$$(55) \qquad 2^{D+4r} + 2^{2r} + 1 = x^2.$$

The left hand side of (55) is quadratic residue (mod 5) if and only if r is even. Put $r = 2k$, ($k \in \mathbb{N}$, $k \geq 1$). Thus

$$(56) \qquad 2^{D+8k} + 2^{4k} + 1 = x^2,$$

which is equivalent to

$$(57) \qquad \frac{x}{2^{\frac{D+8k}{2}}} - 1 = \frac{2^{4k}+1}{2^{\frac{D+8k}{2}}\left(x + 2^{\frac{D+8k}{2}}\right)}.$$

Applying Lemma 2 to the left hand side of (57), and using that (56) gives $2^{\frac{D+8k}{2}} < x$, we obtain

$$(58) \qquad \frac{2^{-43.5}}{2^{(D+8k)\cdot 0.9}} < \frac{2^{4k}+1}{2 \cdot 2^{D+8k}}.$$

We see that $2^{4k} + 1 < 2^{4k+0.5}$ if $k \geq 1$, and by (58) it follows that

$$(59) \qquad D < 32k + 430.$$

On the other hand, considering (56), Lemma 8 provides $D > 56k - 32$, which, together with (59) implies $k \leq 19$. Finally, applying Lemma 1 to (56) with $D_1 = 2^{4k} + 1$, ($k \leq 19$) we conclude that $D \leq 19$, too. A simple computer search shows that equation (56) with odd $D \leq 19$ and $k \leq 19$ has only one solution $D = 1$, $k = 1$, $x = 23$. Hence we obtain the third exceptional solution of (45): $(n, m) = (D + 8k, 4k) = (9, 4)$, and there are no others. So the proof of Theorem 1 is complete.

Proof of Theorem 2. Suppose that $(n, m, x) \in \mathbb{N}^3$ is a positive solution of the diophantine equation

$$(60) \qquad 2^n - 2^m + 1 = x^2.$$

Consider the case $n \geq m$. First let $m \geq 4$. Then (60) is equivalent to the equation

$$(61) \qquad 2^n - D_2 = x^2,$$

where the positive number $D_2 = 2^m - 1$ is odd. By Lemma 3, we find

$$(62) \qquad (n, x) = (m, 1)\ ,\ (2m - 2, 2^{m-1} - 1)$$

as the set of all the solutions of (61) with $m \geq 4$. This result leads to the following solutions of (60):

$$(63) \qquad (n,m,x) = (t,t,1), \quad t \in \mathbb{N},\ t \geq 4;$$

$$(64) \qquad (n,m,x) = (2t, t+1, 2^t - 1), \quad t \in \mathbb{N},\ t \geq 3.$$

The famous case $m = 3$ of (60) has five solutions given by the table in Lemma 3. Among them $(n,m,x) = (3,3,1)$ can be joined to the set (63) with the parameter $t = 3$, moreover, $(n,m,x) = (4,3,3)$ to the set (64) with $t = 2$.

If $m = 2$ or $m = 1$, then Lemma 4 gives the result $(n,m,x) = (2,2,1)$ or $(n,m,x) = (1,1,1)$, respectively. These triplets may be added, for example, to (63) with $t = 2$ and with $t = 1$, respectively.

Finally, it is easy to see that (60) has no solution with $0 < n < m$. Avoiding the repetitions we may summarise the results above as Theorem 2 states.

Proof of Theorem 3. Assume that $(n,m,x) \in \mathbb{N}^3$ with $n \geq m > 0$ is a solution of the equation

$$(65) \qquad 2^n + 2^m - 1 = x^2.$$

If $m \geq 2$, then $2^n + 2^m - 1$ is a quadratic non-residue modulo 4; if $m = 1$, then apply Lemma 4 to have $(n,m,x) = (3,1,3)$.

Now suppose that $(n,m,x) \in \mathbb{N}^3$ is a solution of the equation

$$(66) \qquad 2^n - 2^m - 1 = x^2.$$

Clearly, $n > m$ and $m < 2$. For $m = 1$ apply Lemma 4 to prove the statement.

Proof of Corollary 1. Both sequences G and H have companion polynomial $c(x) = x^2 - 3x + 2$ with zeros $x = 2$ and $x = 1$. It is well known that the terms G_m (and H_m) can be expressed in explicit form. Here by $a_G = G_1 - G_0 = 2^d + 1$ $(a_H = H_1 - H_0 = 2^d - 1)$ and by $b_G = -G_1 + 2G_0 = 1$ $(b_H = -H_1 + 2H_0 = 1)$ we have

$$(67) \qquad G_m = a_G 2^m + b_G = 2^{m+d} + 2^m + 1,$$

$$(68) \qquad H_m = a_H 2^m + b_H = 2^{m+d} - 2^m + 1.$$

Thus to determine all the squares in the recurrences G and H is equivalent to solve the equations (6) and (7) with $n = m + d$ (i.e. $n \geq m$).

Proof of Corollary 2. $\triangle_y = 4^t \triangle_x$ $(y \neq x, y > 0, x > 0)$ implies

$$(69) \qquad y_1^2 - 1 = 4^t (x_1^2 - 1),$$

where $y_1 = 2y + 1 \geq 3$ and $x_1 = 2x + 1 \geq 3$. In virtue of Lemma 5, (69) has no solution under the given conditions.

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(Received July 2001)