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A GEOMETRICAL PROOF OF SUM OF $\cos n\varphi$

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Abstract. In this article, we present a geometrical proof of sum of $\cos \ell\varphi$ where ℓ goes from 1 up to m. Although there exist some summation forms and the proofs are simple, they use complex numbers. Our proof comes from a geometrical construction. Moreover, from this geometrical construction we obtain an other summation form.

1. Introduction

The Lagrange's trigonometric identities are well-known formulas. The one for sum of $\cos \ell\varphi$ ($\varphi \in (0, 2\pi)$) is

$$\sum_{\ell=1}^{m} \cos \ell\varphi = \frac{1}{2}\left(\frac{\sin\,(m+\frac{1}{2})\varphi}{\sin \frac{1}{2}\varphi} - 1\right). \tag{1}$$

The proof of equation (1) is based on the theorem of the complex numbers in all the books, articles and lessons at the universities ([1], [2]). In the following we give a geometrical construction which implies the formula (1) and using certain geometrical properties we obtain an other summation formula without half angles ($\varphi \in (0, 2\pi)$, $\varphi \neq \pi$)

$$\sum_{\ell=1}^{m} \cos \ell\varphi = \frac{1}{2}\left(\frac{\sin(m+1)\varphi + \sin m\varphi}{\sin \varphi} - 1\right). \tag{2}$$

2. Geometrical construction

Let x and e be two lines with the intersection point A_0. Let the angle of them is α as x is rotated to e (Figure 1). Let the point A_1 be given on x such that the distance between the points A_0 and A_1 is 1. Let the point A_2 be on the line e such that the distance of A_1 and A_2 is also equal to 1 and $A_2 \neq A_0$ if $\alpha \neq \pi/2$ and $\alpha \neq 3\pi/2$. Then let the new point A_3 be on the line x again such that $A_2A_3 = 1$ and $A_3 \neq A_1$ if it is possible. Recursively, we can define the point A_ℓ ($\ell \geq 2$) on one of the lines x or e if ℓ is odd or even, respectively, where $A_{\ell-1}A_\ell = 1$ and $A_\ell \neq A_{\ell-2}$ if it is possible. Figure 1 shows the first six points and Figure 2 shows

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some general points. We can easily check that the rotation angels at vertices A_ℓ ($\ell \geq 1$) between the line e (or the axis x) and the segments $A_{\ell-1}A_\ell$ or $A_\ell A_{\ell+1}$ are $(\ell-1)\alpha$ or $(\ell+1)\alpha$, respectively, as the triangles $A_{\ell-1}A_\ell A_{\ell+1}$ are isosceles. (The angle $i\alpha$ can be larger the $\pi/2$, even larger than 2π. The vertices A_ℓ can be closer to A_0 then $A_{\ell-2}$ – see Figure 3.) If A_1 is on the line e we obtain a similar geometric construction. In that case those points A_ℓ are on line x which have even indexes.

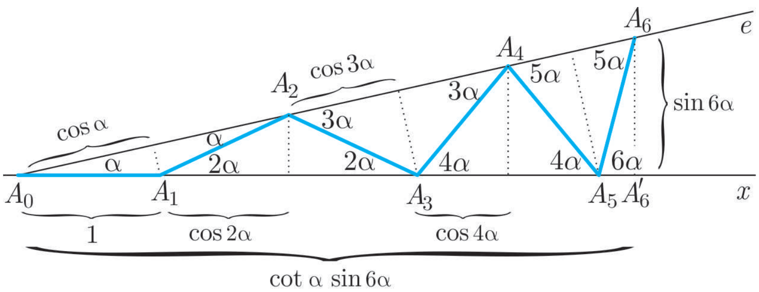

Figure 1. First seven points of the geometrical construction.

Let A_0 be the origin and the line x is the axis x. Then the equation of the line e is $\cos\alpha \cdot y = \sin\alpha \cdot x$. Let A'_n be the orthogonal projection of $A_n \in e$ ($n \geq 2$) onto the axis x then from right angle triangle $A_0A'_nA_n$ the coordinates of the points A_n (see Figure 1) are

$$\begin{aligned} x_n(\alpha) &= \cot\alpha \sin n\alpha, \\ y_n(\alpha) &= \sin n\alpha. \end{aligned} \tag{3}$$

If $\alpha = \pi/2$ and $\alpha = 3\pi/2$ then all the points A_n coincide the points A_0 or A_1, so in the following we exclude this cases.

The parametric equation system of the orbits of the points $A_n \in e$ can be given by the help of the Chebyshev polynomial too (for more details and for some figures of orbits, see in [3]). The equation system is

$$\begin{aligned} x_n(\alpha) &= \cos\alpha\ U_{n-1}(\cos\alpha), \\ y_n(\alpha) &= \sin\alpha\ U_{n-1}(\cos\alpha), \end{aligned} \tag{4}$$

where α goes from 0 to 2π and $U_{n-1}(x)$ is a Chebyshev polynomial of the second kind [4].

Let $A_1 \in x$ and n be even so that $n = 2k+2$. We take the orthogonal projections of the segments $A_{\ell-1}A_\ell$ ($\ell = 1, 2, \ldots, n$) onto the line x (see Figure 1 and 2). Then we realize

$$x_n(\alpha) = 1 + 2\left(\cos 2\alpha + \cos 4\alpha + \cdots + \cos 2k\alpha\right) + \cos(2k+2)\alpha, \tag{5}$$

on the other hand, from (3) or from (4) we have

$$x_n(\alpha) = \cos\alpha\ \tfrac{\sin(2k+2)\alpha}{\sin\alpha}. \tag{6}$$

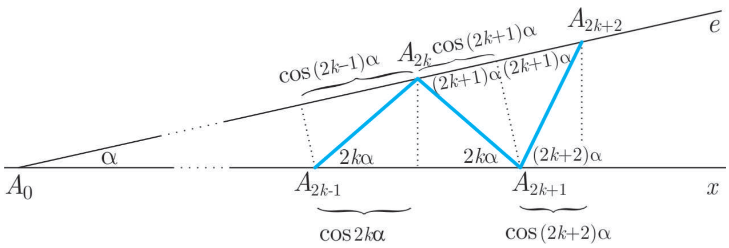

Figure 2. General points of the geometrical construction.

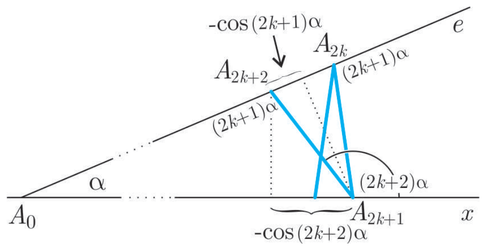

Figure 3. General points.

Comparing (5) and (6) we obtain

$$1 + 2\sum_{\ell=1}^{k} \cos 2\ell\alpha + \cos(2k+2)\alpha = \cos\alpha \, \frac{\sin(2k+2)\alpha}{\sin\alpha}. \tag{7}$$

Using the addition formula for cosine we receive from (7) that

$$\sum_{\ell=1}^{k} \cos 2\ell\alpha = \frac{1}{2}\left(\cos\alpha \, \frac{\sin(2k+2)\alpha}{\sin\alpha} - \cos(2k+2)\alpha - 1\right) = \tag{8a}$$

$$= \frac{1}{2}\left(\frac{\cos\alpha\sin(2k+2)\alpha - \sin\alpha\cos(2k+2)\alpha}{\sin\alpha} - 1\right) = \tag{8b}$$

$$= \frac{1}{2}\left(\frac{\sin((2k+2)-1)\alpha}{\sin\alpha} - 1\right) = \tag{8c}$$

$$= \frac{1}{2}\left(\frac{\sin(2k+1)\alpha}{\sin\alpha} - 1\right). \tag{8d}$$

If $\varphi = 2\alpha$ then

$$\sum_{\ell=1}^{k} \cos \ell\varphi = \frac{1}{2}\left(\frac{\sin(k+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi} - 1\right). \tag{9}$$

3. Other summation form

In this section, we give an other summation form for the cosines without half angles by the help of the defined geometrical construction. Now let us take the orthogonal projection of the segments $A_{\ell-1}A_\ell$ ($\ell = 1, 2, \ldots, n$) onto the line e (see Figure 1 and 2) and summarize them for all ℓ from 1 to $2k+1$. (The sum is equal to $x(\alpha)$ if $n = 2k+1$ and $A_1 \in e$.) Now we gain a similar equation to (7), namely

$$2\left(\cos\alpha + \cos 3\alpha + \cdots + \cos(2k-1)\alpha\right) + \cos(2k+1)\alpha = \cos\alpha\,\tfrac{\sin(2k+1)\alpha}{\sin\alpha}. \tag{10}$$

With analogous calculation to (8) we obtain

$$\sum_{\ell=1}^{k}\cos(2\ell-1)\alpha = \tfrac{1}{2}\left(\cos\alpha\,\tfrac{\sin(2k+1)\alpha}{\sin\alpha} - \cos(2k+1)\alpha\right) = \tag{11a}$$

$$= \tfrac{1}{2}\left(\tfrac{\sin 2k\alpha}{\sin\alpha}\right). \tag{11b}$$

Summing equations (8d) and (11b), we have

$$\sum_{\ell=1}^{k}\cos 2\ell\alpha + \sum_{\ell=1}^{k}\cos(2\ell-1)\alpha = \tfrac{1}{2}\left(\tfrac{\sin(2k+1)\alpha}{\sin\alpha} + \tfrac{\sin 2k\alpha}{\sin\alpha} - 1\right), \tag{12}$$

and finally if $m = 2k$ and $\varphi = \alpha$ we obtain formula (2).

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