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Extending Arithmetic Properties for Compositions Wherein Parts That Are Non-Multiples of *k* Can Be of *t* Kinds

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Abstract

In two papers published in *Quaestiones Mathematicae*, Munagi and Sellers considered the family of functions $D_{k,t}(n)$ that count the number of integer compositions of weight n in which parts not divisible by k can be of t kinds (subscripted $0, 1, \ldots, t-1$). These are also closely related to "inplace" integer compositions of weight n. In their second paper, Munagi and Sellers proved a few divisibility properties satisfied by $D_{k,t}(n)$ for specific values of k, t, and n. In this work, we significantly extend these arithmetic properties, providing infinitely families of congruences. Our proof techniques are quite elementary, relying on the structure of the generating functions in question.

Keywords Composition · Congruence · Inplace · Generating function · Recurrence

MSC Classification 05A15 · 11P83

1 Introduction

A partition of a positive integer n is a sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Each λ_i, $1 \leq i \leq r$, is called a part of the partition, while n is often referred to as the weight of the partition. More information about integer partitions can be found in [1] and [2].

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A composition of a positive integer n is a sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. (That is, a composition of n is a partition of n wherein all permutations of the parts are allowed.) It is well–known that the number of compositions of weight n, which we denote by $c(n)$, is given by $c(n) = 2^{n-1}$. For example, the compositions of weight $n = 3$ are given by

$$3, \quad 2+1, \quad 1+2, \quad 1+1+1,$$

so that $c(3) = 2^{3-1} = 4$. Additional information about compositions, including results for a variety of restricted compositions, can be found in [4].

In [5] and [6], Munagi and Sellers defined the family of functions $D_{k,t}(n)$ that count the number of integer compositions of weight n in which parts not divisible by k can be of t kinds (subscripted $0, 1, \ldots, t-1$). For example, $D_{2,2}(3) = 14$ thanks to the following compositions:

$$\begin{aligned}
&3_0, \quad 3_1, \quad 2+1_0, \quad 2+1_1, \quad 1_0+2, \quad 1_1+2,\\
&1_0+1_0+1_0, \quad 1_0+1_1+1_0, \quad 1_0+1_0+1_1, \quad 1_0+1_1+1_1,\\
&1_1+1_0+1_0, \quad 1_1+1_1+1_0, \quad 1_1+1_0+1_1, \quad 1_1+1_1+1_1.
\end{aligned}$$

The sequence of values $D_{2,2}(n)$ appears in [7, A052945]. Moreover, the values of $D_{2,2}(n)$ and $D_{2,3}(n)$ appear in [8, Table 1].

Munagi and Sellers proved that, for any k and t, $D_{k,t}(n)$ satisfies the initial conditions

$$\begin{aligned}
&D_{k,t}(0) = 1,\\
&D_{k,t}(i) = t(t+1)^{i-1} \text{ for } 1 \le i \le k-1, \text{ and}\\
&D_{k,t}(k) = t(t+1)^{k-1} - (t-1).
\end{aligned}$$

Moreover, for $n > k$, $D_{k,t}(n)$ satisfies the recurrence

$$D_{k,t}(n) = t(D_{k,t}(n-1) + \cdots + D_{k,t}(n-(k-1))) + 2D_{k,t}(n-k).$$

Equivalently, the generating function for $D_{k,t}(n)$ is given by

$$\mathfrak{D}_{k,t}(x) := \sum_{n \ge 0} D_{k,t}(n)x^n = \frac{1-x^k}{1-2x^k - t\sum_{i=1}^{k-1} x^i}. \tag{1}$$

In [6], Munagi and Sellers proved a number of arithmetic properties satisfied by $D_{k,t}(n)$ based on (1) above, as well as the corresponding recurrence relation and initial conditions satisfied by $D_{k,t}(n)$. These arithmetic properties include the following:

Theorem 1.1 *For $n > 0$, $n \not\equiv 0 \pmod{k}$, $D_{k,t}(n) \equiv 0 \pmod{t}$.*

Theorem 1.2 *We have the following:*

- *If $t \equiv 1 \pmod 2$, then $D_{k,t}(n) \equiv 0 \pmod 2$ for all $n \ne 0, 1$.*
- *If $t \equiv 0 \pmod 2$, then $D_{k,t}(n) \equiv 0 \pmod 2$ for all $n \ne 0, k$.*

At the end of [6], the authors share the following: "Additional results for moduli that are higher powers of 2 appear to also be true, especially for values of t which are congruent to 1 modulo larger powers of 2. We leave the discovery and proof of such results to the reader."

Our initial goal in this brief note is to generalize Theorem 1.2 extensively by finding an infinite family of values of k and t such that congruences like those above hold modulo arbitrarily large powers of 2. Once this goal is met, we will then turn to studying new congruences for the functions $D_{k,t}(n)$ modulo other primes (for specific values k and t).

In order to complete some of the generating function manipulations above, we utilize the following straightforward fact from differential calculus:

Lemma 1.3 *Let $f^{(n)}(x)$ denote the n^{th} derivative of $f(x)$ with respect to x where n is a positive integer. Then*

$$\left(\frac{1}{1-x}\right)^{(n)} = \frac{n!}{(1-x)^{n+1}}.$$

With the above in hand, we now transition to proving several new arithmetic properties.

2 New arithmetic properties for $D_{2,t}(n)$

We begin our discussion by focusing on congruences modulo large powers of 2 satisfied by a family of functions wherein $k = 2$.

Theorem 2.1 *For all $s \geq 1$ and all $a \geq 1$, $\mathfrak{D}_{2,a2^s+1}(x) \equiv 1 + x + 2x^2 + 4x^3 + \cdots + 2^{s-1}x^s \pmod{2^s}$.*

Proof We begin by noting that

$$\begin{aligned}
&\mathfrak{D}_{2,a2^s+1}(x)\\
&= \frac{1-x^2}{1-2x^2-(a2^s+1)x}\\
&= (1-x^2)\sum_{j\geq 0}(2x^2+(a2^s+1)x)^j\\
&\equiv (1-x^2)\sum_{j\geq 0}(2x^2+x)^j \pmod{2^s}\\
&= (1-x^2)\sum_{j\geq 0}x^j(2x+1)^j\\
&= (1-x^2)\sum_{j\geq 0}x^j\sum_{i=0}^{j}\binom{j}{i}2^ix^i\\
&\equiv (1-x^2)\sum_{j\geq 0}x^j\left(1+\binom{j}{1}2x+\binom{j}{2}2^2x^2+\cdots+\binom{j}{s-1}2^{s-1}x^{s-1}\right) \pmod{2^s}\\
&= (1-x^2)\left(\sum_{j\geq 0}x^j+2x^2\sum_{j\geq 0}\binom{j}{1}x^{j-1}+4x^4\sum_{j\geq 0}\binom{j}{2}x^{j-2}+\cdots+2^{s-1}x^{s-1}\sum_{j\geq 0}\binom{j}{s-1}x^{j-(s-1)}\right)\\
&= (1-x^2)\left(\left(\frac{1}{1-x}\right)+2x^2\left(\frac{1}{1-x}\right)^{(1)}+\frac{4x^4}{2!}\left(\frac{1}{1-x}\right)^{(2)}+\cdots+\frac{2^{s-1}x^{2(s-1)}}{(s-1)!}\left(\frac{1}{1-x}\right)^{(s-1)}\right).
\end{aligned}$$

Thanks to Lemma 1.3, this means

$$\begin{aligned}
&\mathfrak{D}_{2,a2^s+1}(x)\\
&\quad\equiv (1-x^2)\left(\frac{1}{1-x}+2x^2\frac{1!}{(1-x)^2}+\frac{4x^4}{2!}\frac{2!}{(1-x)^3}+\cdots+\frac{2^{s-1}x^{2(s-1)}}{(s-1)!}\frac{(s-1)!}{(1-x)^s}\right) \pmod{2^s}\\
&\quad= \frac{(1-x^2)}{(1-x)^s}\left((1-x)^{s-1}+(2x^2)(1-x)^{s-2}+(2x^2)^2(1-x)^{s-3}+\cdots+(2x^2)^{s-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-x^2)}{(1-x)^s} \cdot (1-x)^{s-1} \left(1 + \frac{2x^2}{1-x} + \frac{(2x^2)^2}{(1-x)^2} + \cdots + \frac{(2x^2)^{s-1}}{(1-x)^{s-1}}\right) \\
&= \frac{(1-x^2)}{(1-x)} \left(1 + \frac{2x^2}{1-x} + \left(\frac{2x^2}{1-x}\right)^2 + \cdots + \left(\frac{2x^2}{1-x}\right)^{s-1}\right) \\
&= \frac{(1-x^2)}{(1-x)} \cdot \frac{\left(\frac{2x^2}{1-x}\right)^s - 1}{\frac{2x^2}{1-x} - 1} \\
&= \frac{(1-x^2)}{(1-x)} \cdot \frac{\left(\frac{2x^2}{1-x}\right)^s - 1}{\frac{2x^2+x-1}{1-x}} \\
&= (1-x^2) \cdot \frac{\left(\frac{2x^2}{1-x}\right)^s - 1}{2x^2+x-1} \\
&= (1-x^2) \cdot \frac{\left(\frac{2x^2}{1-x}\right)^s - 1}{(2x-1)(x+1)} \\
&= (1-x^2) \cdot \frac{(2x^2)^s - (1-x)^s}{(1-x)^s(2x-1)(x+1)} \\
&\equiv (1-x^2) \cdot \frac{-(1-x)^s}{(1-x)^s(2x-1)(x+1)} \pmod{2^s} \\
&\qquad \text{since } (2x^2)^s = 2^s x^{2s} \text{ has coefficient divisible by } 2^s \text{ when } s \geq 1 \\
&= (1-x^2) \cdot \frac{1}{(1-2x)(1+x)} \\
&= \frac{1-x}{1-2x},
\end{aligned}
$$

and it is easy to show that this rational function gives

$$
1 + x + 2x^2 + 4x^3 + 8x^4 + \cdots = 1 + \sum_{n \geq 1} 2^{n-1} x^n.
$$

The result follows. □

As an extension of Theorem 1.2 above, note the following corollary of the above theorem.

Corollary 2.2 *For all $s \geq 1$ and all $a \geq 1$, $D_{2,a2^s+1}(n) \equiv 0 \pmod{2^s}$ for all $n \geq 2^{s-1}$.*

The first bullet point in Theorem 1.2 is clearly related to the above (considering $s = 1$).

We next prove an infinite family of congruences modulo p where $p \equiv 1$ or $3 \pmod 8$; indeed, there are two such Ramanujan-like congruences for every such prime.

Theorem 2.3 *Let $p > 3$ be prime, $p \equiv 1$ or $3 \pmod 8$. Let c be an integer, $1 \leq c \leq p-1$, such that $c^2 \equiv -2 \pmod p$, and let $t = 2c$. Finally, let r be defined by*

$$
r = \begin{cases} 2k, & \text{if } p = 6k+1 \text{ for some integer } k \\ p - 2k, & \text{if } p = 6k-1 \text{ for some integer } k \end{cases}.
$$

Then, for all integers $n \geq 0$, $D_{2,t}(pn + r) \equiv 0 \pmod p$.

Proof We begin by noting that, since $p \equiv 1 \text{ or } 3 \pmod 8$, we know that -2 is a quadratic residue modulo p. Thus, the value c defined above exists. Next, note that

$$\begin{aligned}\sum_{n\geq 0} D_{2,t}(n)q^n &= \frac{1-x^2}{1-2x^2-tx} \\ &\equiv \frac{1-x^2}{1+c^2x^2-tx} \pmod p \quad \text{since } c^2 \equiv -2 \pmod p \\ &= \frac{1-x^2}{(1-cx)^2} \quad \text{since } t \equiv 2c \\ &= (1-x^2)\sum_{k\geq 0}(k+1)(cx)^k.\end{aligned}$$

Note that we can provide a second branch, when the denominator is given by $1+c^2x^2-tx = (1+cx)^2$ with $t \equiv -2c$, the negative of $2c$. This is incongruent to $t = 2c$ modulo p, because if $2c \equiv -2c$, then $4c \equiv 0$, and then $c \equiv 0$, a contradiction. Corollary 2.4 below will describe this case.

By reading off the coefficients of the series in the expression above, this means that, for $n \geq 2$,

$$\begin{aligned}D_{2,t}(n) &\equiv (n+1)c^n - (n-1)c^{n-2} \pmod p \\ &\equiv c^{n-2}(c^2(n+1)-(n-1)) \pmod p \\ &\equiv c^{n-2}((c^2-1)n + (c^2+1)) \pmod p.\end{aligned}$$

Now we restrict our attention to $D_{2,t}(pn+r)$ with r defined above. Then we see that

$$\begin{aligned}D_{2,t}(pn+r) &\equiv c^{r-2}((c^2-1)r + (c^2+1)) \pmod p \\ &\equiv c^{r-2}(-3r-1) \pmod p \quad \text{since } c^2 \equiv -2 \pmod p \\ &= -c^{r-2}(3r+1).\end{aligned}$$

Now we quickly consider our two cases:

Case 1: $p = 6k+1$, so that $r = 2k$ or $r = (p-1)/3$. Then

$$\begin{aligned}D_{2,t}(pn+r) &\equiv -c^{r-2}(3r+1) \pmod p \\ &= -c^{r-2}\left(3\cdot\frac{p-1}{3}+1\right) \\ &\equiv 0 \pmod p\end{aligned}$$

which is the desired result.

Case 2: $p = 6k-1$, so that $r = p-2k$ or $r = (2p-1)/3$. Then

$$\begin{aligned}D_{2,t}(pn+r) &\equiv -c^{r-2}(3r+1) \pmod p \\ &= -c^{r-2}\left(3\cdot\frac{2p-1}{3}+1\right) \\ &\equiv 0 \pmod p\end{aligned}$$

which is the desired result. □

The first few congruences identified in the theorem above are the following:

$$D_{2,6}(11n+7) \equiv 0 \pmod{11},$$

$$\begin{aligned} D_{2,14}(17n+11) &\equiv 0 \pmod{17}, \\ D_{2,12}(19n+6) &\equiv 0 \pmod{19}, \\ D_{2,22}(41n+27) &\equiv 0 \pmod{41}, \\ D_{2,32}(43n+14) &\equiv 0 \pmod{43}. \end{aligned}$$

Given the description in Theorem 2.3 we see that, for each congruence listed above, the value of t is even (since $t = 2c$ and c is an integer). It is important to note that, for each congruence found in Theorem 2.3, there is a related congruence that is easily described as follows:

Corollary 2.4 *Given all the assumptions in Theorem 2.3, we have, for all* $n \geq 0$,

$$D_{2,p-t}(pn+r) \equiv 0 \pmod{p}.$$

Proof Notice that

$$\begin{aligned} \sum_{n\geq 0} D_{2,p-t}(n)q^n &= \frac{1-x^2}{1-2x^2-(p-t)x} \\ &\equiv \frac{1-x^2}{1+c^2x^2+tx} \pmod{p} \\ &= \frac{1-x^2}{(1+cx)^2} \quad \text{since } t = 2c \\ &= (1-x^2)\sum_{k\geq 0}(k+1)(-cx)^k. \end{aligned}$$

and the rest of the proof follows as before. □

Thus, for example, the congruences mentioned above have the following companion congruences:

$$\begin{aligned} D_{2,5}(11n+7) &\equiv 0 \pmod{11}, \\ D_{2,3}(17n+11) &\equiv 0 \pmod{17}, \\ D_{2,7}(19n+6) &\equiv 0 \pmod{19}, \\ D_{2,19}(41n+27) &\equiv 0 \pmod{41}, \\ D_{2,11}(43n+14) &\equiv 0 \pmod{43}. \end{aligned}$$

In addition to the mod p congruences that are proven in Theorem 2.3 and Corollary 2.4, where one must restrict to the case where $p \equiv 1 \text{ or } 3 \pmod{8}$ and the congruences hold at arithmetic progressions of the form $pn + r$, it is the case that additional congruences hold for numerous primes, regardless of their least nonnegative residue modulo 8, where the arithmetic progressions are of the form $(p-1)n + r$ and $(p+1)n + r$, respectively. We focus our attention on these results now. Interestingly, these results hinge on the periodicity modulo p of the values of $D_{2,t}(n)$. In this vein, note the following:

Theorem 2.5 *Let* p *be an odd prime and let* t, $1 \leq t \leq p-1$, *be such that* $\left(\frac{t^2+8}{p}\right) = 1$, *where* $\left(\frac{a}{p}\right)$ *is the usual Legendre symbol. Then, for all* $n > 0$, $D_{2,t}(n) \equiv D_{2,t}(n+p-1) \pmod{p}$.

Proof The proof of this result follows immediately from the work of Gupta, Rockstroh, and Su [3, Theorem 6]. □

Thanks to the above periodicity result modulo p, we can immediately state the following corollary which provides us with numerous Ramanujan-like congruences satisfied by the functions $D_{2,t}(n)$.

Corollary 2.6 *Let p be an odd prime and let t, $1 \leq t \leq p-1$, be such that $\left(\frac{t^2+8}{p}\right) = 1$. Also, let r, $1 \leq r \leq p-1$ be such that $D_{2,t}(r) \equiv 0 \pmod{p}$. Then, for all $n \geq 0$, $D_{2,t}((p-1)n+r) \equiv 0 \pmod{p}$.*

An example may be extremely helpful at this time. Let $p = 23$. Note that, for $t = 4$, we have

$$\left(\frac{4^2+8}{23}\right) = \left(\frac{24}{23}\right) = \left(\frac{1}{23}\right) = 1$$

from the standard properties of the Legendre symbol. Thus, we now have a candidate for a congruence of the form

$$D_{2,4}(22n+r) \equiv 0 \pmod{23}$$

for all $n \geq 0$ if we can find a single value of r, $1 \leq r \leq 22$, such that $D_{2,4}(r) \equiv 0 \pmod{23}$. Indeed, note that $D_{2,4}(15) = 4574546176 \equiv 0 \pmod{23}$, and this provides the ingredient needed to finalize this example. In fact, following a similar approach to the above, we have the following congruences modulo $p = 23$: For all $n \geq 0$,

$$\begin{aligned}
D_{2,2}(22n+10) &\equiv 0 \pmod{23},\\
D_{2,4}(22n+15) &\equiv 0 \pmod{23},\\
D_{2,8}(22n+13) &\equiv 0 \pmod{23},\\
D_{2,10}(22n+17) &\equiv 0 \pmod{23},\\
D_{2,13}(22n+17) &\equiv 0 \pmod{23},\\
D_{2,15}(22n+13) &\equiv 0 \pmod{23},\\
D_{2,19}(22n+15) &\equiv 0 \pmod{23},\\
D_{2,21}(22n+10) &\equiv 0 \pmod{23}.
\end{aligned}$$

3 Closing thoughts

3.1 Additional results

We begin this section by highlighting an interesting infinite family of congruences modulo 3 that occurs at $t = 3\ell + 2$.

Theorem 3.1 *For all $k \geq 3$ and for all $\ell \geq 0$,*

$$\sum_{n\geq 0} D_{k,3\ell+2}(n)x^2 \equiv 1 + 2\left(\sum_{j\geq 0} x^{(k+1)j+1} + \sum_{j\geq 0} x^{(k+1)j+k} + \sum_{j\geq 0} x^{(k+1)j+(k+1)}\right) \pmod{3}.$$

Before proving Theorem 3.1, we note that this provides a characterization of the function $D_{k,3\ell+2}(n)$ modulo 3 for any $n \geq 0$. In particular, we immediately have the following corollary.

Corollary 3.2 *For all $n \geq 0$, $\ell \geq 0$, and $2 \leq r \leq k-1$,*

$$D_{k,3\ell+2}((k+1)n+r) \equiv 0 \pmod{3}.$$

Proof of Theorem 3.1 Thanks to the generating function (1), we know

$$\begin{aligned}
\sum_{n\geq 0} D_{k,3\ell+2}(n)x^n &= \frac{1-x^k}{1-2x^k-(3\ell+2)\sum_{i=1}^{k-1}x^i} \\
&= \frac{1-x^k}{1-(3\ell+2)\sum_{i=1}^{k}x^i} \\
&= \frac{1-x^k}{1-\left((3\ell+2)\sum_{i=0}^{k}x^i\right)+3\ell+2} \\
&\equiv \frac{1-x^k}{\sum_{i=0}^{k}x^i} \pmod 3 \\
&= \frac{1-x^k}{\left(\frac{1-x^{k+1}}{1-x}\right)} \\
&= \frac{(1-x^k)(1-x)}{1-x^{k+1}} \\
&= (1-x-x^k+x^{k+1})\sum_{j\geq 0}\left(x^{k+1}\right)^j \\
&= 1+\sum_{j\geq 0}x^{(k+1)j+(k+1)}-\sum_{j\geq 0}x^{(k+1)j+1}-\sum_{j\geq 0}x^{(k+1)j+k}+\sum_{j\geq 0}x^{(k+1)j+(k+1)} \\
&\equiv 1+2\sum_{j\geq 0}x^{(k+1)j+1}+2\sum_{j\geq 0}x^{(k+1)j+k}+2\sum_{j\geq 0}x^{(k+1)j+(k+1)} \pmod 3.
\end{aligned}$$

□

In addition, an overarching principle exists which allows us to naturally extend a single congruence satisfied by the function $D_{k,t}(n)$ into an infinite family of related results. The key theorem is the following:

Theorem 3.3 *For all $n \geq 0$, and for any modulus m,*

$$D_{k,t}(n) \equiv D_{k,t+m}(n) \pmod m.$$

The proof of this result is immediate given the generating function result (1) above.

Proof We have

$$\begin{aligned}
\sum_{n\geq 0} D_{k,t+m}(n)x^n &= \frac{1-x^k}{1-2x^k-(t+m)\sum_{i=1}^{k-1}x^i} \\
&\equiv \frac{1-x^k}{1-2x^k-t\sum_{i=1}^{k-1}x^i} \pmod m \\
&= \sum_{n\geq 0} D_{k,t}(n)x^n.
\end{aligned}$$

□

3.2 Computational observations and open questions

We close this portion of our dialogue by noting that, for various primes p, our computations indicate that there are numerous additional congruences modulo p satisfied by $D_{2,t}((p+$

$1)n + r)$ where t satisfies $\left(\frac{t^2+8}{p}\right) = -1$. These congruences appear to be closely linked to a periodicity result which is related to that in Theorem 2.5 above. (See [3, Theorem 8] for additional details.) Returning to the prime $p = 23$, it appears that the following hold for all n:

$$\begin{aligned}
D_{2,5}(24n + 22) &\equiv 0 \pmod{23},\\
D_{2,6}(24n + 6) &\equiv 0 \pmod{23},\\
D_{2,7}(24n + 8) &\equiv 0 \pmod{23},\\
D_{2,9}(24n + 9) &\equiv 0 \pmod{23},\\
D_{2,14}(24n + 9) &\equiv 0 \pmod{23},\\
D_{2,16}(24n + 8) &\equiv 0 \pmod{23},\\
D_{2,17}(24n + 6) &\equiv 0 \pmod{23},\\
D_{2,18}(24n + 22) &\equiv 0 \pmod{23}.
\end{aligned}$$

We leave it to the interested reader to complete the argument necessary to prove a general result of this form which is complementary to Corollary 2.6 above.

In conclusion, we see that we have greatly generalized the congruence results (modulo 2 and modulo t) which were proven by Munagi and Sellers [6]. Of course, it would be nice to see combinatorial proofs of the above divisibility properties. We leave this as a possibility for interested readers.

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