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ON TWO-VARIABLE RECURRENCE SEQUENCES WITH A TILING APPLICATION

László Németh*

Abstract. In discrete mathematics, recursive systems of equations occur frequently. For this reason, in this article we focus specifically on a special case of these: a type of nested two-variable recursive systems. We develop a method by which these systems can be transformed into equivalent recursive relations containing a single sequence. As an application, we analyze a tiling problem arising on a $2 \times n$ grid and derive the corresponding recursive relation. The example illustrates the efficiency and simplicity of the proposed method.

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1. Introduction

Especially in the field of combinatorics, researchers frequently encounter not only recursive sequences and equations, but also systems of equations consisting of recursive equations. It is essential to find methods for solving them, where by "solutions" we mean recursive equations expressed solely in terms of their own elements.

One of the simplest and most common systems of equations consists of nested linear equations. Many publications deal with their solution, for example, in [1], p. 33 and [2, 3]. Another type involves nested equations that refer back to previous variables, not just one step, but possibly several. Németh and Szalay [4] explicitly derived the coefficients of the recursive equation from the solution by using the matrix theory in the case of two equations, and the authors presented a general algorithmic solution for more equations in [5]. We also note that Belbachir and Bencherif [6] introduced the general case of recurrent linear sequences with one index.

As our main aim in this article, we present a solution for a system of recurrence equations in which the recurrences have two variables. For example, the following system of two-variable recurrence equations, which arises in the tiling problem defined in this article, is

$$
\begin{aligned}
r_n^b &= r_{n-1}^b + r_{n-2}^{b-2} + c_n^b + c_n^{b-1} + c_{n-1}^{b-1} + c_{n-1}^{b-2},\\
c_n^b &= r_{n-1}^b + r_{n-1}^{b-1} + r_{n-2}^{b-1} + r_{n-2}^{b-2} + c_{n-2}^{b-2}.
\end{aligned}
$$

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University of Sopron, Institute of Basic Sciences, Bajcsy Zs. u. 4., 9400 Sopron, Hungary.

* Corresponding author: nemeth.laszlo@uni-sopron.hu

for $n, b > 2$, and the solution for the term r_n^b follows as a corollary of our main theorem discussed later, namely, for $n, b \geq 4$ that

$$r_n^b = 2\,r_{n-1}^b + 2\,r_{n-1}^{b-1} + r_{n-1}^{b-2} + 2\,r_{n-2}^{b-1} + 6\,r_{n-2}^{b-2} + 2\,r_{n-2}^{b-3} + 2\,r_{n-3}^{b-3} + r_{n-3}^{b-4} - r_{n-4}^{b-4}.$$

Organization of the paper. Section 2 presents the general case and its solution: we define a system of equations using a two-part, two-variable recursive sequence and two equations, then provide a solution for one of the sequences; we express the recursion using only its own elements. The method we present is somewhat similar to the elimination method for solving systems of linear equations. We then extend our method to multiple sequences. Section 3 addresses a tiling problem as an example: we define a not-too-complex tiling problem on a $2 \times n$ grid, for which we specify the number of possible cases using a system of recursive equations. We use the result obtained earlier to arrive at the final solution. We also highlight some connections among the previous results. Furthermore, we note that our tiling problem could be solved using other methods, but we specifically chose this problem and solution method to provide an easily understandable, illustrative example of the main result of our article.

2. System of two-variable recurrence sequences

In this section, we define a special type of recursive equation system consisting of nested two-variable recursive sequences. Let u and v be positive integers, and let the sequences $(a_{n,m})$ and $(b_{n,m})$ be given for $n \geq u$ and $m \geq v$, by the system

$$\begin{aligned}
0 &= \alpha_{0,0}^1 a_{n,m} + \alpha_{0,1}^1 a_{n,m-1} + \alpha_{1,0}^1 a_{n-1,m} + \alpha_{1,1}^1 a_{n-1,m-1} + \cdots + \alpha_{u,v}^1 a_{n-u,m-v} \\
&\quad + \beta_{0,0}^1 b_{n,m} + \beta_{0,1}^1 b_{n,m-1} + \beta_{1,0}^1 b_{n-1,m} + \beta_{1,1}^1 b_{n-1,m-1} + \cdots + \beta_{u,v}^1 b_{n-u,m-v}, \\
0 &= \alpha_{0,0}^2 a_{n,m} + \alpha_{0,1}^2 a_{n,m-1} + \alpha_{1,0}^2 a_{n-1,m} + \alpha_{1,1}^2 a_{n-1,m-1} + \cdots + \alpha_{u,v}^2 a_{n-u,m-v} \\
&\quad + \beta_{0,0}^2 b_{n,m} + \beta_{0,1}^2 b_{n,m-1} + \beta_{1,0}^2 b_{n-1,m} + \beta_{1,1}^2 b_{n-1,m-1} + \cdots + \beta_{u,v}^2 b_{n-u,m-v},
\end{aligned} \tag{2.1}$$

where at least one of the coefficients α's or β's with index u ($\alpha_{u,\cdot}^{\cdot}$ or $\beta_{u,\cdot}^{\cdot}$) is nonzero, and similarly, at least one of $\alpha_{\cdot,v}^{\cdot}$ or $\beta_{\cdot,v}^{\cdot}$ is nonzero. Shortly,

$$\begin{aligned}
\sum_{i=0}^{u}\sum_{j=0}^{v} \alpha_{i,j}^1 a_{n-i,m-j} &= -\sum_{i=0}^{u}\sum_{j=0}^{v} \beta_{i,j}^1 b_{n-i,m-j}, \\
\sum_{i=0}^{u}\sum_{j=0}^{v} \alpha_{i,j}^2 a_{n-i,m-j} &= -\sum_{i=0}^{u}\sum_{j=0}^{v} \beta_{i,j}^2 b_{n-i,m-j}.
\end{aligned} \tag{2.2}$$

where $\alpha_{i,j}^p$ and $\beta_{i,j}^p$ are real numbers, the equations are independent, and the orders in variables n and m are u and v, respectively.

Theorem 2.1. *The recurrence sequence $(a_{n,m})$ given by the system of recurrence equations (2.1) can be described for $n \geq 2u$ and $m \geq 2v$, by the recurrence form*

$$\sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v} \left(\beta_{k,\ell}^2 \alpha_{i,j}^1 - \beta_{k,\ell}^1 \alpha_{i,j}^2\right) a_{n-i-k,m-j-\ell} = 0,$$

where its orders in variables n and m are at most $2u$ and $2v$, respectively.

We have introduced a much shorter notation using the following symbols. Let

$$\begin{aligned}
A_{n,m}^{(1)} &= \sum_{i=0}^{u}\sum_{j=0}^{v} \alpha_{i,j}^1 a_{n-i,m-j}, \quad & B_{n,m}^{(1)} &= \sum_{i=0}^{u}\sum_{j=0}^{v} \beta_{i,j}^1 b_{n-i,m-j}, \\
A_{n,m}^{(2)} &= \sum_{i=0}^{u}\sum_{j=0}^{v} \alpha_{i,j}^2 a_{n-i,m-j}, \quad & B_{n,m}^{(2)} &= \sum_{i=0}^{u}\sum_{j=0}^{v} \beta_{i,j}^2 b_{n-i,m-j}.
\end{aligned}$$

This yields the system of equations (2.2) in the form

$$\begin{aligned} A^{(1)}_{n,m} &= -B^{(1)}_{n,m}, \\ A^{(2)}_{n,m} &= -B^{(2)}_{n,m}. \end{aligned} \tag{2.3}$$

Now, for example, if we shift the left-hand side of the first equation by -2 and -3 in the variables n and m, respectively, we obtain

$$\begin{aligned} A^{(1)}_{n-2,m-3} &= \sum_{i=0}^{u}\sum_{j=0}^{v}\alpha^1_{i,j}a_{n-2-i,m-3-j} \\ &= \alpha^1_{0,0}a_{n-2,m-3} + \alpha^1_{0,1}a_{n-2,m-4} + \alpha^1_{1,0}a_{n-3,m-3} + \cdots + \alpha^1_{u,v}a_{n-u-2,m-v-3}. \end{aligned}$$

We will eliminate the terms of the sequence $(b_{n,m})$ from (2.1). To do this, we expand the system of equations by shifting the equations in both the n and m variables. Let the shifts be $0 \le k \le u$ and $0 \le \ell \le v$, respectively, and multiply them by the unknowns $\Phi^1_{k,\ell}$ and $\Phi^2_{k,\ell}$, respectively. This gives us $2(u+1)\cdot(v+1)$ equations. Depending on the coefficients, we may obtain related, equivalent, or even identically zero equations. Next, we determine the Φ unknowns such that the sum of the right-hand sides is zero. Thus, the sum of the left-hand sides will be precisely the recursion of the sequence $(a_{n,m})$. To determine all $\Phi^1_{k,\ell}$ and $\Phi^2_{k,\ell}$, we must solve the homogeneous linear system of equations formed from the right-hand sides, where the variables are the Φ's. Furthermore, it is unnecessary to use a shift greater than u and v; we would not obtain any more independent equations, and since the system of linear equations is homogeneous, we are looking for only one non-trivial solution among the many possible solutions.

The following lemma establishes that $\Phi^1_{k,\ell} = \beta^2_{k,\ell}$ and $\Phi^2_{k,\ell} = \beta^1_{k,\ell}$ for any k and ℓ.

Lemma 2.2. *For the terms of the sequence $(b_{n,m})$, the following equation holds:*

$$\sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^2_{k,\ell}B^{(1)}_{n-k,m-\ell} = \sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^1_{k,\ell}B^{(2)}_{n-k,m-\ell}.$$

Proof. The proof follows from the commutativity of real multiplication ($\beta^1_{k,\ell}\beta^2_{i,j} = \beta^2_{i,j}\beta^1_{k,\ell}$). Thus,

$$\begin{aligned} \sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^2_{k,\ell}B^{(1)}_{n-k,m-\ell} &= \sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^2_{k,\ell}\sum_{i=0}^{u}\sum_{j=0}^{v}\beta^1_{i,j}b_{n-i-k,m-j-\ell} \\ &= \sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v}\beta^2_{k,\ell}\beta^1_{i,j}b_{n-i-k,m-j-\ell} \\ &= \sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v}\beta^1_{k,\ell}\beta^2_{i,j}b_{n-i-k,m-j-\ell} \\ &= \sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^1_{k,\ell}\sum_{i=0}^{u}\sum_{j=0}^{v}\beta^2_{i,j}b_{n-i-k,m-j-\ell} \\ &= \sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^1_{k,\ell}B^{(2)}_{n-k,m-\ell}. \end{aligned}$$

□

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. By applying the previously mentioned shift extension to the system of equations (2.3), we obtain

$$\begin{aligned}\sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^2_{k,\ell}A^{(1)}_{n-k,m-\ell}&=-\sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^2_{k,\ell}B^{(1)}_{n-k,m-\ell},\\ \sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^1_{k,\ell}A^{(2)}_{n-k,m-\ell}&=-\sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^1_{k,\ell}B^{(2)}_{n-k,m-\ell}.\end{aligned}$$

Since, by Lemma 2.2, the right-hand sides are equal, for the left-hand sides, we obtain

$$\begin{aligned}0&=\sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^2_{k,\ell}A^{(1)}_{n-k,m-\ell}-\sum_{k=0}^{u}\sum_{\ell=0}^{v}\beta^1_{k,\ell}A^{(2)}_{n-k,m-\ell}\\ &=\sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v}\beta^2_{k,\ell}\alpha^1_{i,j}\ a_{n-i-k,m-j-\ell}-\sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v}\beta^1_{k,\ell}\alpha^2_{i,j}a_{n-i-k,m-j-\ell}\\ &=\sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v}\left(\beta^2_{k,\ell}\alpha^1_{i,j}-\beta^1_{k,\ell}\alpha^2_{i,j}\right)a_{n-i-k,m-j-\ell}.\end{aligned}$$

□

2.1. System of multi-variable recurrence sequences

In this subsection, we generalize our result to the case of a system of equations consisting of several – say, $s\geq 2$ – different recursive sequences and independent lines. Let the system of equations be given by

$$\begin{aligned}0&=\sum_{q=1}^{s}\sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{q,(1)}_{i,j}r^q_{n-i,m-j},\\ 0&=\sum_{q=1}^{s}\sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{q,(2)}_{i,j}r^q_{n-i,m-j},\\ &\vdots\\ 0&=\sum_{q=1}^{s}\sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{q,(s)}_{i,j}r^q_{n-i,m-j},\end{aligned}\tag{2.4}$$

in more compact form it is

$$0=\sum_{q=1}^{s}\sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{q,(p)}_{i,j}r^q_{n-i,m-j},\quad p=1,\dots,s.$$

If $s=2$, the result is (2.1) with $\alpha^p_{i,j}=\gamma^{1,(p)}_{i,j}$, $\beta^p_{i,j}=\gamma^{2,(p)}_{i,j}$, $a_{n,m}=r^1_{n,m}$, and $b_{n,m}=r^2_{n,m}$.

Unfortunately, describing the general solution – for example, for the sequence $(r^1_{n,m})$ – would require a very complex formula; instead, we will present the method that everyone already suspects after the case where $s=2$. The possible algorithm for determining the recurrence sequence $(r^1_{n,m})$ from the system of equations (2.4) is as follows: First, we eliminate the last $(r^s_{n,m})$ terms from the first two equations, obtaining a new equation as a result. Next we eliminate $(r^s_{n,m})$ from the old first and third equations, then from the old first and fourth equations and so on, finally from the old first and last equations. Each step is completely analogous to the case when $s = 2$. This gives us a system of equations consisting of an $s-1$ sequence and $s-1$ equations. Next, we similarly eliminate the terms $(r^q_{n,m})$ for q, one by one from $s-1$ to 2. Finally, we obtain the recursion for $(r^1_{n,m})$. Algorithm 1 provides its precise description. Note that there may be a faster algorithm that requires fewer computational steps.

Algorithm 1 Systematic elimination for sequence $(r^1_{n,m})$

```
Input: System of equations S = {E_1, E_2, ..., E_s}
for q = s down to 2 do
    for k = 2 to q do
        Eliminate r^q_{n,m} using E_1 and E_k → E'_{1,k}
    end for
    S ← {E'_{1,2}, ..., E'_{1,q}}
end for
Output: Recurrence relation for (r^1_{n,m})
```

Below, we briefly outline the algorithm for the case where $s = 3$. Now, our system of equations is

$$\begin{aligned}
0 &= \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{1,(1)}_{i,j}r^1_{n-i,m-j} + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{2,(1)}_{i,j}r^2_{n-i,m-j} + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{3,(1)}_{i,j}r^3_{n-i,m-j},\\
0 &= \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{1,(2)}_{i,j}r^1_{n-i,m-j} + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{2,(2)}_{i,j}r^2_{n-i,m-j} + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{3,(2)}_{i,j}r^3_{n-i,m-j},\\
0 &= \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{1,(3)}_{i,j}r^1_{n-i,m-j} + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{2,(3)}_{i,j}r^2_{n-i,m-j} + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{3,(3)}_{i,j}r^3_{n-i,m-j}.
\end{aligned}$$

Converted to form

$$\begin{aligned}
0 &= \sum_{i=0}^{u}\sum_{j=0}^{v}\left(\gamma^{1,(1)}_{i,j}r^1_{n-i,m-j} + \gamma^{2,(1)}_{i,j}r^2_{n-i,m-j}\right) + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{3,(1)}_{i,j}r^3_{n-i,m-j},\\
0 &= \sum_{i=0}^{u}\sum_{j=0}^{v}\left(\gamma^{1,(2)}_{i,j}r^1_{n-i,m-j} + \gamma^{2,(2)}_{i,j}r^2_{n-i,m-j}\right) + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{3,(2)}_{i,j}r^3_{n-i,m-j},\\
0 &= \sum_{i=0}^{u}\sum_{j=0}^{v}\left(\gamma^{1,(3)}_{i,j}r^1_{n-i,m-j} + \gamma^{2,(3)}_{i,j}r^2_{n-i,m-j}\right) + \sum_{i=0}^{u}\sum_{j=0}^{v}\gamma^{3,(3)}_{i,j}r^3_{n-i,m-j}.
\end{aligned}$$

Firstly, from the first two lines and the first and last lines, using Theorem 2.1 we eliminate all the terms of sequence $(r^3_{n,m})$, then we have

$$0 = \sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v} \gamma^{3,(2)}_{k,\ell}\left(\gamma^{1,(1)}_{i,j} r^1_{n-i-k,m-j-\ell} + \gamma^{2,(1)}_{i,j} r^2_{n-i-k,m-j-\ell}\right) - \gamma^{3,(1)}_{k,\ell}\left(\gamma^{1,(2)}_{i,j} r^1_{n-i-k,m-j-\ell} + \gamma^{2,(2)}_{i,j} r^2_{n-i-k,m-j-\ell}\right),$$

$$0 = \sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v} \gamma^{3,(3)}_{k,\ell}\left(\gamma^{1,(1)}_{i,j} r^1_{n-i-k,m-j-\ell} + \gamma^{2,(1)}_{i,j} r^2_{n-i-k,m-j-\ell}\right) - \gamma^{3,(1)}_{k,\ell}\left(\gamma^{1,(3)}_{i,j} r^1_{n-i-k,m-j-\ell} + \gamma^{2,(3)}_{i,j} r^2_{n-i-k,m-j-\ell}\right).$$

Thus,

$$0 = \sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v} \left(\gamma^{3,(2)}_{k,\ell}\gamma^{1,(1)}_{i,j} - \gamma^{3,(1)}_{k,\ell}\gamma^{1,(2)}_{i,j}\right) r^1_{n-i-k,m-j-\ell} + \left(\gamma^{3,(2)}_{k,\ell}\gamma^{2,(1)}_{i,j} - \gamma^{3,(1)}_{k,\ell}\gamma^{2,(2)}_{i,j}\right) r^2_{n-i-k,m-j-\ell},$$

$$0 = \sum_{k=0}^{u}\sum_{\ell=0}^{v}\sum_{i=0}^{u}\sum_{j=0}^{v} \left(\gamma^{3,(3)}_{k,\ell}\gamma^{1,(1)}_{i,j} - \gamma^{3,(1)}_{k,\ell}\gamma^{1,(3)}_{i,j}\right) r^1_{n-i-k,m-j-\ell} + \left(\gamma^{3,(3)}_{k,\ell}\gamma^{2,(1)}_{i,j} - \gamma^{3,(1)}_{k,\ell}\gamma^{2,(3)}_{i,j}\right) r^2_{n-i-k,m-j-\ell}.$$

Secondly, we eliminate the terms of $(r^2_{n,m})$, then

$$0 = \sum_{i,k,i^*,k^*=0}^{u}\ \sum_{j,\ell,j^*,\ell^*=0}^{v} (X - Y)\, r^1_{n-i-k-i^*-k^*,m-j-\ell-j^*-\ell^*},$$

where

$$X = \left(\gamma^{3,(3)}_{k^*,\ell^*}\gamma^{2,(1)}_{i^*,j^*} - \gamma^{3,(1)}_{k^*,\ell^*}\gamma^{2,(3)}_{i^*,j^*}\right)\left(\gamma^{3,(2)}_{k,\ell}\gamma^{1,(1)}_{i,j} - \gamma^{3,(1)}_{k,\ell}\gamma^{1,(2)}_{i,j}\right),$$
$$Y = \left(\gamma^{3,(2)}_{k^*,\ell^*}\gamma^{2,(1)}_{i^*,j^*} - \gamma^{3,(1)}_{k^*,\ell^*}\gamma^{2,(2)}_{i^*,j^*}\right)\left(\gamma^{3,(3)}_{k,\ell}\gamma^{1,(1)}_{i,j} - \gamma^{3,(1)}_{k,\ell}\gamma^{1,(3)}_{i,j}\right).$$

3. Examples

3.1. A tiling example

The tilings with (1×1)-squares and (1×2)-dominoes (two squares with a common edge) are a very familiar topic among researchers in discrete mathematics. Hereafter, a square always means a (1×1)-square, and a domino means a (1×2)-domino. It is known, *e.g.*, in [7], that the number of possible tilings of a $(1 \times n)$-board is F_{n+1}, where (F_n) denotes the Fibonacci sequence (defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ and A000045 in OEIS [8]). McQuistan and Lichtman [9] (generalizations in [10]) studied the $(2 \times n)$-board tilings with squares and dominoes in the case of Euclidean square mosaic, and they proved that the sequence (r_n) of the number of possible tilings satisfies the third-order recurrence relation $r_n = 3r_{n-1} + r_{n-2} - r_{n-3}$ (A030186). Some researchers generalized the tiling with colored shapes [11–13]. Moreover, Komatsu, Németh, and Szalay [14] examined the tilings with colored squares and dominoes on the hyperbolic $(2 \times n)$-board, and they gave the

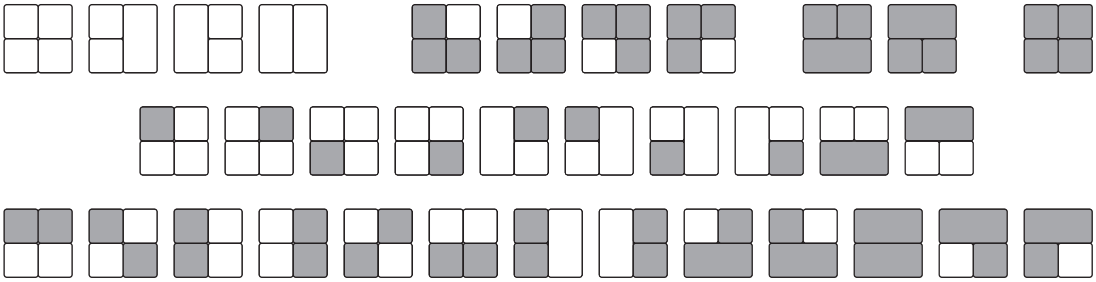

FIGURE 1. Tilings on (2×2)-board.

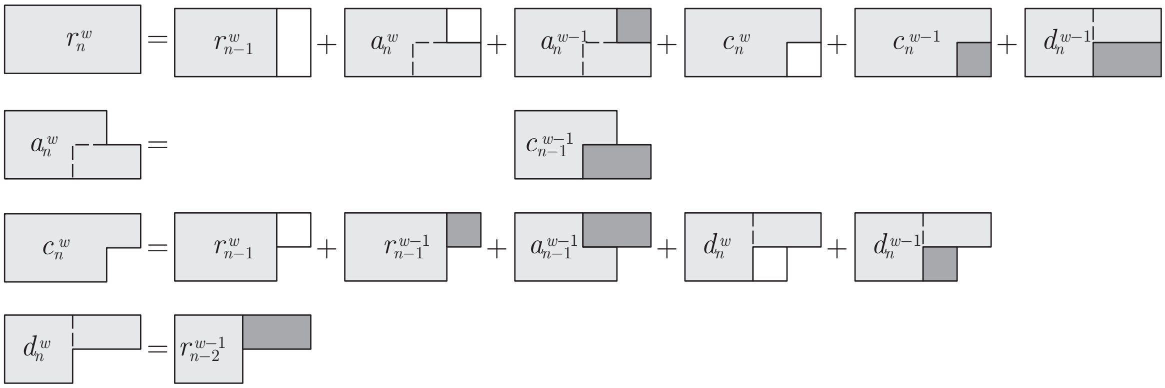

FIGURE 2. Tiling with squares and dominoes on a $(2 \times n)$-board.

fourth order linear homogeneous recurrence relation of r_n, where the coefficients satisfy recurrence sequences, as well.

In this subsection, we present a simple tiling problem, using the method described earlier as an example. Let r_n^b be the number of the different tilings on a $(2 \times n)$-board with black and white squares and dominoes such that the black dominoes can only be placed horizontally, while the white ones can only be placed vertically, and the number of all black tiles (squares and dominoes) in a tiling is b. For example, Figure 1 shows all the possible tilings on (2×2)-board. Thus, $r_2^0 = 4$, $r_2^1 = 10$, $r_2^2 = 13$, $r_2^3 = 6$, and $r_2^4 = 1$. Moreover, for better distinguishability, we write the number of black tiles as a superscript.

We distinguish three different types of not finished tilings on a $(2 \times n)$-board. According to the first column of Figure 2, type A is when the last lower tile is a domino and the last upper tile, the square, is missing; type C is when the last lower square is missing; type D is when the last upper tile is a domino and the last lower tile, the domino, is missing. Let a_n, c_n, and d_n be the numbers of their elements, respectively. If we put tiles into the missing parts, then we can build our tilings recursively. For example, in the third row, we can get tilings of type C by putting squares and dominoes from full-, type A, or type D tilings.

Figure 2 shows how we can obtain the tilings recursively, which is given in the following lemma.

Lemma 3.1. *For $n \geq 2$, the tiling sequence (r_n^b) is recursively defined by the system of recurrence*

$$\begin{aligned} r_n^b &= r_{n-1}^b + a_n^b + a_n^{b-1} + c_n^b + c_n^{b-1} + d_n^{b-1}, \\ a_n^b &= c_{n-1}^{b-1}, \\ c_n^b &= r_{n-1}^b + r_{n-1}^{b-1} + a_{n-1}^{b-1} + d_n^b + d_n^{b-1}, \\ d_n^b &= r_{n-2}^{b-1}, \end{aligned} \tag{3.1}$$

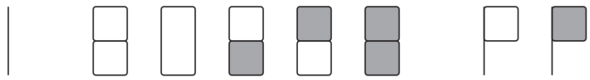

FIGURE 3. Initial terms of tiling with squares and dominoes.

					1					
				2	2	1				
			4	10	13	6	1			
		8	32	66	68	38	10	1		
	16	88	248	398	393	232	79	14	1	
32	224	800	1728	2448	2298	1423	564	136	18	1

FIGURE 4. Triangle $\mathcal{T}$.

	w	$w-1$	$w-2$	$w-3$	$w-4$
n	-1				
$n-1$	2	2	1		
$n-2$		2	6	2	
$n-3$				2	1
$n-4$					-1

FIGURE 5. Coefficients of the recurrence of sequence (r_n^b).

where the initial values are $r_0^0 = 1$, $r_1^0 = 2$, $r_1^1 = 2$, $r_1^2 = 1$, $c_1^0 = 1$, $c_1^1 = 1$ *(see Fig. 3), and additionally, all the others are zero.*

Using the system (3.1) we can arrange the terms of sequence (r_n^b) into a triangle form as Figure 4 shows the first few terms. The bth entry of row n is equal to r_n^b.

Theorem 3.2. *The tiling sequence* (r_n^b) *is recursively given by the two-variables constant-coefficients homogeneous linear recurrence relation for* $n, b \geq 4$,

$$\begin{aligned} r_n^b = {} & 2\,r_{n-1}^b + 2\,r_{n-1}^{b-1} + r_{n-1}^{b-2} + 2\,r_{n-2}^{b-1} + 6\,r_{n-2}^{b-2} + 2\,r_{n-2}^{b-3} \\ & + 2\,r_{n-3}^{b-3} + r_{n-3}^{b-4} - r_{n-4}^{b-4}, \end{aligned} \tag{3.2}$$

where the initial values are in the first four rows of triangle $\mathcal{T}$ *in Figure 4 when the bth entry of row n is equal to* r_n^b*. Additionally,* $r_n^b = 0$ *for* $n < 0$*, or* $b > 2n + 1$ *or* $b < 0$*.*
The coefficients of the recurrence of sequence (r_n^b) *set equal to* 0 *can be seen in a table form in Figure 5.*

Proof. The terms of sequences (a_n^b) and (d_n^b) can easily be eliminated from the system of recurrence equations (3.1). We substitute $a_n^b = c_{n-1}^{b-1}$, $a_n^{b-1} = c_{n-1}^{b-2}$, and $d_n^{b-1} = r_{n-2}^{b-2}$ into the first row, and similar shifted items into

$$
\begin{array}{c|ccc}
\alpha^1_{i,j} & w & w{-}1 & w{-}2 \\ \hline
n & 1 & & \\
n{-}1 & -1 & & \\
n{-}2 & & & -1
\end{array}
\qquad
\begin{array}{c|ccc}
\alpha^2_{i,j} & w & w{-}1 & w{-}2 \\ \hline
 & & & \\
 & 1 & 1 & \\
 & & 1 & 1
\end{array}
\qquad
\begin{array}{c|ccc}
\beta^1_{i,j} & w & w{-}1 & w{-}2 \\ \hline
 & -1 & -1 & \\
 & & -1 & -1 \\
 & & &
\end{array}
\qquad
\begin{array}{c|ccc}
\beta^2_{i,j} & w & w{-}1 & w{-}2 \\ \hline
 & -1 & & \\
 & & & \\
 & & & 1
\end{array}
$$

FIGURE 6. Values of $\alpha^1_{i,j}$, $\alpha^2_{i,j}$, $\beta^1_{i,j}$, and $\beta^2_{i,j}$.

the third row. Then we obtain

$$
\begin{aligned}
r^b_n &= r^b_{n-1} + r^{b-2}_{n-2} + c^b_n + c^{b-1}_n + c^{b-1}_{n-1} + c^{b-2}_{n-1}, \\
c^b_n &= r^b_{n-1} + r^{b-1}_{n-1} + r^{b-1}_{n-2} + r^{b-2}_{n-2} + c^{b-2}_{n-2}.
\end{aligned}
\tag{3.3}
$$

By arranging it into form (2.3), it is

$$
\begin{aligned}
r^b_n - r^b_{n-1} - r^{b-2}_{n-2} &= c^b_n + c^{b-1}_n + c^{b-1}_{n-1} + c^{b-2}_{n-1}, \\
r^b_{n-1} + r^{b-1}_{n-1} + r^{b-1}_{n-2} + r^{b-2}_{n-2} &= c^b_n - c^{b-2}_{n-2}.
\end{aligned}
$$

Now, we consider Theorem 2.1. The sequences (r^b_n) and (c^b_n) play the roles of $(a_{n,m})$ and $(b_{n,m})$. We find that $u = v = 2$ and the values of $\alpha^1_{i,j}$, $\alpha^2_{i,j}$, $\beta^1_{i,j}$, and $\beta^2_{i,j}$ are presented left to right, respectively, in Figure 6.

Thus, for recurrence sequence (r^b_n), we have

$$
\sum_{k=0}^{2}\sum_{\ell=0}^{2}\sum_{i=0}^{2}\sum_{j=0}^{2}\left(\beta^2_{k,\ell}\alpha^1_{i,j} - \beta^1_{k,\ell}\alpha^2_{i,j}\right) r^{b-j-\ell}_{n-i-k} = 0,
$$

which happens to be exactly recursions (3.2). We calculated its coefficients by computer. □

We give some connections between our tiling results and earlier works. Let s_n denote the sum of the nth row of triangle $\mathcal{T}$. Then the sequence (s_n) counts all tilings with squares and dominoes on a $(2 \times n)$-board with squares of two colors and dominoes of only one color. The colors of the dominoes become irrelevant. Moreover,

$$
(s_n)_{n\geq 0} = \{1, 5, 34, 223, 1469, 9672, 63685, 419329, \ldots\} = \texttt{A102436},
$$

and from [12] we know that $s_n = 6s_{n-1} + 4s_{n-2} - s_{n-3}$ for $n \geq 3$.

3.2. An earlier unsolved system of recurrences

In the article [15], the authors did not manage to solve a similar system of recurrence sequences. They dealt with a self-avoiding walks combinatorial problem on a rectangular grid, and for one of their results, they got a system of recurrence equations with defined five sequences (for more details, see Thm. 9 and its proof in [15]). In fact, their final solution was the following system

$$
\begin{aligned}
b^w_n &= 2b^w_{n-1} - b^w_{n-2} + 2c^{w-1}_{n-1} - 2c^{w-1}_{n-2} - b^{w-1}_{n-2} + b^{w-1}_{n-3} + 2c^{w-2}_{n-3}, \\
c^w_n &= b^w_n + c^w_{n-1} - b^{w-1}_{n-2} - c^{w-1}_{n-2}.
\end{aligned}
\tag{3.4}
$$

They should have given the recurrence of the sequence (c_n^w) with variables n and w. Now, we are lucky to solve it because of our useful tools, the Theorem 2.1. For this, we rearrange (3.4) into

$$\begin{aligned} 2c_{n-1}^{w-1} - 2c_{n-2}^{w-1} + 2c_{n-3}^{w-2} &= b_n^w - 2b_{n-1}^w + b_{n-2}^w + b_{n-2}^{w-1} - b_{n-3}^{w-1}, \\ c_n^w - c_{n-1}^w + c_{n-2}^{w-1} &= b_n^w - b_{n-2}^{w-1}. \end{aligned}$$

and using our theorem for $n \geq 5$ and $w \geq 3$, we obtain

$$c_n^w = 3c_{n-1}^w + 2c_{n-1}^{w-1} - 3c_{n-2}^w - 4c_{n-2}^{w-1} + c_{n-3}^w + 4c_{n-3}^{w-1} - 2c_{n-4}^{w-1} + c_{n-4}^{w-2} + c_{n-5}^{w-2} - 2c_{n-5}^{w-3}.$$

We also note that further investigations into self-avoiding walks on finite grids can lead to even more complex systems. Furthermore, Belbachir *et al.* [16] recently formulated and solved a system of three-variable recurrence equations arising from step-constrained self-avoiding walks, providing an explicit solution for a specific case.

4. Conclusion and future works

In this article, we defined a system of equations with two variables, consisting of nested recursive sequences – first two, then more – and provided a possible solution method for it. The solution means that each sequence is defined solely in terms of its own previous terms. As an example, we defined a tiling problem in which we were able to successfully apply the result of our main theorem. Furthermore, we solved a system of equations that arose in a previous article, which the authors were only able to express in a simplified form. They were unable to eliminate the terms of the other sequences.

We believe that our result will be useful to researchers who need to simplify and solve systems of equations consisting of complex recursive sequences. Our method can easily be adapted to similar problems.

For certain values, it is conceivable that there is a recursion of lower order than what we generally obtain as a result. For example, the method can also be used to solve systems of recursive equations with a single variable, in which case $v = 0$. In the article [17], the author solved a system of equations defined by two recursive sequences and obtained a 9th-order recursion for one of the sequences, whereas with the method discussed in this article, we obtain an extended version of this, an 11th-order recursive equation. It would be of interest to determine under what conditions the resulting recurrence has minimal order. In particular, when the orders of the recursions in the system of equations do not match.

In connection with the tiling problem, one can examine the number of white dominoes instead of black ones, or even both at the same time. In the latter case, we would obtain a system of equations consisting of three-variable recursive sequences. The elements of the solution sequence could be visualized not as triangles, but as tetrahedrons.

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