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Exploring Fibonacci and Padovan diagonals in the Motzkin triangle

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Abstract The Motzkin triangle is considered as the zero-free part of a well-defined plane array. The right diagonal leg of the triangle is the Motzkin sequence, which satisfies a second order linear recurrence with linear polynomial coefficients.

We give some new formulas for the items of the Motzkin triangle. Then we determine a recursion for the sum of entries lying along the so-called Fibonacci and Padovan diagonals in the Motzkin triangle.

Keywords Motzkin triangle · linear recurrence · Fibonacci diagonal · Padovan diagonal · bi^{s}nomial coefficient

Mathematics Subject Classification (2010) 05A19 · 11B37 · 05A17 · 11C08

1 Introduction

Define the array $[m_{n,k}]$ for $n,k \in \mathbb{Z}$, as follows. Let $m_{n,k} = 0$ unless $0 \le k \le n$. In the latter case, $m_{0,0} = 1$, and for $n \ge 1$ the elements $m_{n,k}$ satisfy

$$m_{n,k} = m_{n-1,k} + m_{n-1,k-1} + m_{n-1,k-2}.$$

The zero-free part of the array has a triangular shape structurally identical to Pascal's triangle, and it is called Motzkin triangle (A026300, and reflected version A064189 in OEIS [9], it is an arithmetical triangle working with the rule of bi^{2}nomial coefficients triangle). The name comes from the fact that the right leg of the triangle coincides with the Motzkin sequence $(M_n)_{n\ge0}$ (i.e., $m_{n,n} = M_n$) defined by $M_0 = M_1 = 1$ and for $n \ge 2$,

$$M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}.$$

It is known that the Motzkin sequence satisfies the D-finite linear recurrence (see [7, Chapter 7])

$$(n+2)M_n = (2n+1)M_{n-1} + (3n-3)M_{n-2}. \tag{1.1}$$

Clearly, the left leg is the constant 1 sequence, i.e., $m_{n,0} = 1$. Moreover, it is easy to see that $m_{n,1} = n$ fulfills for any n. Figure 1 illustrates the Motzkin triangle together with the construction rule, and the Motzkin sequence as well.

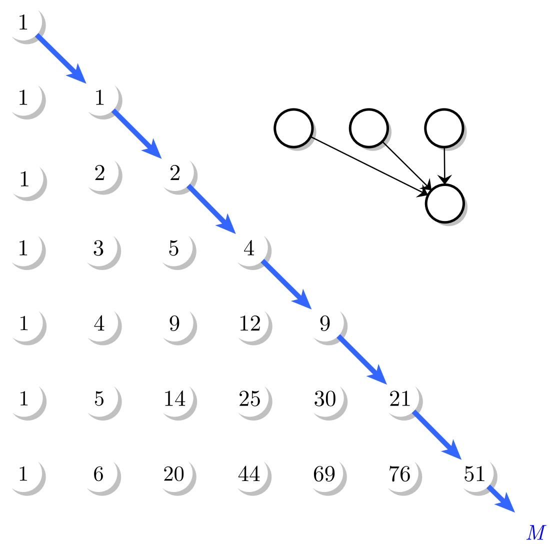

Fig. 1 Motzkin triangle — A026300

The work [8] was devoted to investigating various configurations in the Motkin triangle. One of the main results of that paper is Theorem 1.1, which provides a linear recursive relationship among three consecutive entries of a row in the triangle. It shows that two consecutive elements of a row are enough to construct all the entries of the row.

Theorem 1.1 *Let $n \in \mathbb{N}$, and assume that $k \le n+1$. Then the recurrence*

$$k(n-k+3)m_{n,k} = (n-k+1)(n-k+3)m_{n,k-1} + (n-k+1)(2n-k+4)m_{n,k-2} \tag{1.2}$$

holds in the rows of the Motzkin triangle.

We even see in [8] that the entries of a column in the Motzkin triangle are polynomial values such that the degree of polynomial $p_k(n)$ belonging to column k is precisely k. For example, $p_0(n) = 1$, $p_1(n) = n$, $p_2(n) =$

$(n-1)(n+2)/2$, etc. If one considers n as a variable in (1.2) and replaces $m_{n,k}$ by $p_k(n)$, then (1.2) provides exactly the sequence of polynomials $(p_k(n))$.

In this section, we even recall the notion of trinomial coefficients. Note that sometimes $\binom{n}{k}_2$ (see later) is called a trinomial coefficient, but in this paper we always use the following definition: These are the coefficients in $(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,k_2,k_3} x^{k_1} y^{k_2} z^{k_3}$ given by

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1! \cdot k_2! \cdot k_3!},$$

where the conditions $0 \le k_1, k_2, k_3 \le n$ hold. For $\binom{n}{k}_2$, we use the phrase bi^{2}nomial numbers or bi^{2}nomial coefficients, which can be given combinatorically as follows: $\binom{n}{k}_2$ is the number of distinct ways to distribute k uniform objects in n labeled boxes such that each box may contain at most 2 objects. There are fewer works studying the numbers $\binom{n}{k}_2$ (or generally the bi^{s}nomial numbers $\binom{n}{k}_s$, see [4], [5], and [6] for further properties and details). Probably, the reason is that there is no nice simple formula for them. One form can be obtained easily if the expansion of the term $(1+x+x^2)^n$ is considered by the polynomial theorem (as above). Indeed, if the exponent $k = k_2 + 2k_3$ is fixed, then knowing $k_1 + k_2 + k_3 = n$ we have the consequence

$$\binom{n}{k}_2 = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2j, j, n-k+j},$$

a connection between the bi^{2}nomial coefficients $\binom{n}{k}_2$ and the trinomial coefficients. For an analogous version, see [3].

In this paper, we reveal some other phenomena related to the Motzkin triangle. Basically, we concentrate on the sums of the entries located along some specific rays.

2 Results

The properties of the Motzkin triangle were examined in [8]. Here we study some more features related to the triangle.

2.1 Certain formulae and row sums

The first result (Theorem 2.1) is mentioned in [9] at A026300, but we have not found any reference for the proof. So we present a proof, and provide the reverse identity in Corollary 2.2.

Theorem 2.1 *For all $0 \le k \le n$, we have*

$$m_{n,k} = \binom{n}{k}_2 - \binom{n}{k-2}_2.$$

Corollary 2.2 *Assume $0 \leq k \leq n$. Then*

$$\binom{n}{k}_2 = \sum_{j=0}^{\lfloor k/2 \rfloor} m_{n,k-2j}.$$

The following two theorems and the corollary of the first one investigate problems related to row sums.

Theorem 2.3 *For $0 \leq k \leq n$, we have*

$$\sum_{j=0}^{k} m_{n,j} = \binom{n}{k-1}_2 + \binom{n}{k}_2.$$

In particular, the row-sums in the Motzkin triangle satisfy

$$\sum_{j=0}^{n} m_{n,j} = \binom{n}{n-1}_2 + \binom{n}{n}_2.$$

Corollary 2.4 *Assume $0 \leq k \leq n$. Then*

$$\sum_{j=0}^{k} p_j(n) = \binom{n}{k-1}_2 + \binom{n}{k}_2.$$

Theorem 2.5 *For the alternating sums, we find*

$$\sum_{j=0}^{n} (-1)^j m_{n,j} = (-1)^n \left(\binom{n}{n-1}_2 - \binom{n}{n}_2 \right).$$

2.2 Fibonacci direction

In Pascal's triangle, the ascending diagonal sums establish the Fibonacci sequence, which is a binary recurrence. Thus, the direction which results the Fibonacci sequence is often called Fibonacci direction. The analogous problem in the Motzkin triangle leads to a D-finite linear recurrence of order 7. Not like Pascal's triangle, the Motzkin triangle does not possess a vertical symmetry. Therefore, it has a sense to consider the symmetric pair of the Fibonacci direction, the result will be described in Theorem 2.7.

Theorem 2.6 *The ascending diagonal sum*

$$s_n := \sum_{j=0}^{\lfloor n/2 \rfloor} m_{n-j,j}$$

is D-linear recurrent sequence of order 7 given by

$$\begin{aligned}(n+1)s_n = (n-1)s_{n-1} + 3(n-1)s_{n-2} - (n-5)s_{n-3} + (n-11)s_{n-4} \\ - (5n-19)s_{n-5} - 3(n-5)s_{n-6} - 3(n-5)s_{n-7}.\end{aligned}$$

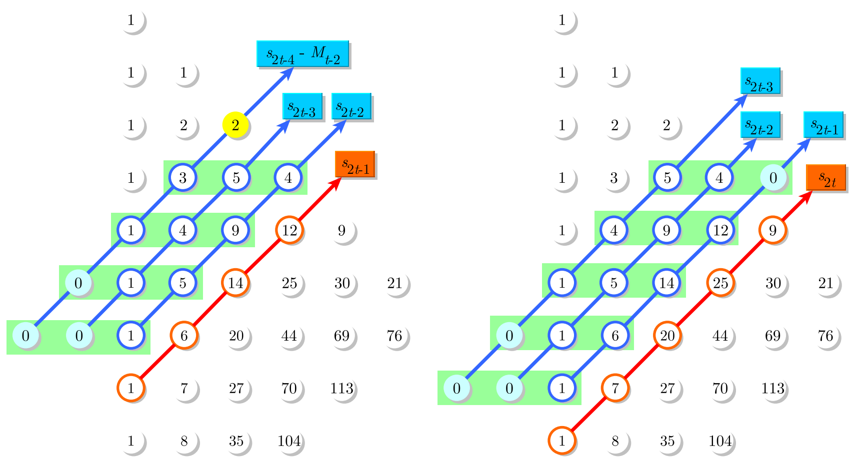

Fig. 2 Fibonacci ascending direction in Motzkin triangle

Theorem 2.7 *The descending diagonal sum*

$$r_n := \sum_{j=0}^{\lfloor n/2 \rfloor} m_{n-j,n-2j}$$

can be given as a D-linear recurrent sequence of order 3 by

$$(n+3)r_n = nr_{n-1} + (5n+3)r_{n-2} + 3nr_{n-3}.$$

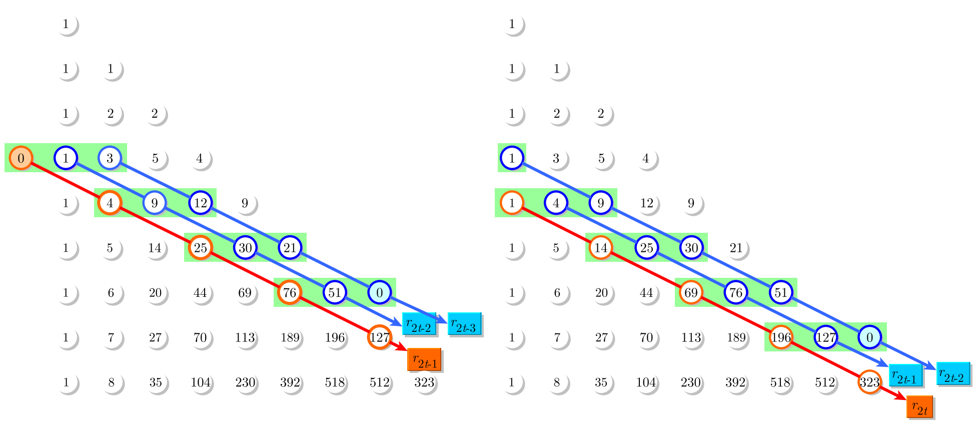

Fig. 3 Fibonacci descending direction in Motzkin triangle

2.3 Padovan direction

In Pascal's triangle, the Padovan direction was named in [1,2]. Considering the sums of the elements of Pascal's triangle, whose direction is parallel to the vector $(2,1)$ (two steps to the right and one step up in the triangle), we obtain the Padovan numbers. We can call this direction the Padovan direction. Since the structure of Pascal's and Motzkin triangles are identical, we can adopt the Padovan direction here. Beginning at $m_{n,0}$, the corresponding ray of the Padovan ascending direction crosses the elements $m_{n-1,2}, m_{n-2,4}, \dots$, their sum is denoted by $p_n := \sum_{j=0}^{\lfloor n/3 \rfloor} m_{n-j,2j}$ (see Figure 4). The intermediate sums between p_n and p_{n+1} is given by $q_n := \sum_{j=0}^{\lfloor (n-1)/3 \rfloor} m_{n-j,1+2j}$. One claims naturally the description of these two sequences. The associate problem stands for the Padovan descending direction, provides the two sequences

$$u_n = \sum_{j=0}^{\lfloor n/3 \rfloor} m_{n-j,n-2j}, \quad \text{and} \quad v_n = \sum_{j=0}^{\lfloor (n-1)/3 \rfloor} m_{n-j,n-1-2j}.$$

The ascending and descending Padovan problem can be handled by the same method, and the final results are quite complicated. Therefore here we deal with the easier one, namely the descending direction. A note will indicate the other question after the end of the proof of the first statement (Theorem 2.8).

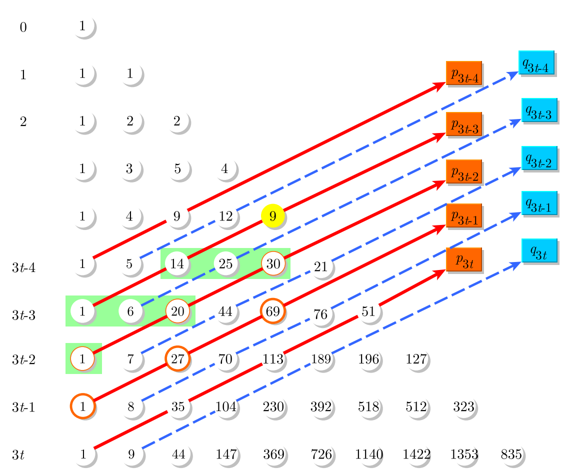

Fig. 4 Padovan ascending direction in Motzkin triangle

Theorem 2.8 *For $n \geq 0$, the terms of sequences (u_n) and (v_n) satisfy the recurrences*

$$
\begin{aligned}
(t+1)u_t = (5t+3)u_{t-1} - (2t+8)u_{t-2} - (10t-28)u_{t-3} - (5t-9)u_{t-4} \\
- (3t-9)u_{t-5},
\end{aligned}
$$

and

$$
\begin{aligned}
(t^2-4)v_t = (5t^2-3t-11)v_{t-1} - (2t^2+1)v_{t-2} - (10t^2-30t+23)v_{t-3} \\
- (5t^2-12t+7)v_{t-4} - (3t^2-9t+6)v_{t-5},
\end{aligned}
$$

respectively.

3 Proofs

Proof of Theorem 2.1. We apply the technique of induction for the rows of the Motzkin triangle.

Clearly, one can check that the identity is valid for the small values $n = 0, 1, 2$, for example,

$$
\begin{aligned}
m_{2,0} &= 1 = 1 - 0 = \binom{2}{0}_2 - \binom{2}{-2}_2, \\
m_{2,1} &= 2 = 2 - 0 = \binom{2}{1}_2 - \binom{2}{-1}_2, \\
m_{2,2} &= 2 = 3 - 1 = \binom{2}{2}_2 - \binom{2}{0}_2.
\end{aligned}
$$

Now assume that the statement holds for row $(n-1)$. Then, using the construction rule of the Motzkin triangle and the hypothesis, we have

$$
\begin{aligned}
m_{n,k} &= m_{n-1,k-2} + m_{n-1,k-1} + m_{n-1,k} \\
&= \left(\tbinom{n-1}{k-2}_2 - \tbinom{n-1}{k-4}_2\right) + \left(\tbinom{n-1}{k-1}_2 - \tbinom{n-1}{k-3}_2\right) + \left(\tbinom{n-1}{k}_2 - \tbinom{n-1}{k-2}_2\right) \\
&= \left(\tbinom{n-1}{k-2}_2 + \tbinom{n-1}{k-1}_2 + \tbinom{n-1}{k}_2\right) - \left(\tbinom{n-1}{k-4}_2 + \tbinom{n-1}{k-3}_2 + \tbinom{n-1}{k-2}_2\right) \\
&= \tbinom{n}{k}_2 - \tbinom{n}{k-2}_2.
\end{aligned}
$$

□

Proof of Corollary 2.2. Insert expression (2.2) into $\sum_{j=0}^{\lfloor k/2 \rfloor} m_{n,k-2j}$. It becomes a telescopic sum, and finally we arrive at $\binom{n}{k}_2 - 0$.

□

Proof of Theorem 2.3. The proof is a direct application of the previous theorem. Indeed,

$$\sum_{j=0}^{k} m_{n,j} = \sum_{j=0}^{k} \left(\binom{n}{j}_2 - \binom{n}{j-2}_2 \right)$$
$$= -\binom{n}{-2}_2 - \binom{n}{-1}_2 + \binom{n}{k-1}_2 + \binom{n}{k}_2 .$$

Since the first two remaining terms are zero, then the proof of the first part is complete. The result is obviously true if $k = n$, and this gives the row-sum formula.

□

Proof of Corollary 2.4. The corollary is trivial.

□

Proof of Theorem 2.5. Following the idea of the proof of Theorem 2.3, we need only one more observation, namely the term $\binom{n}{k-2}$ vanishes in the partial sum

$$\pm \left(\binom{n}{k}_2 - \binom{n}{k-2}_2 \right) \pm \left(\binom{n}{k-2}_2 - \binom{n}{k-4}_2 \right).$$

□

Proof of Theorem 2.6. At the beginning, we distinguish two cases according to the parity of n, see Figure 2.

Case n is odd. Now $[n/2] = (n-1)/2$. Then we have

$$s_n = \sum_{k=0}^{(n-1)/2} m_{n-k,k} = \sum_{k=0}^{n/2} (m_{n-1-k,k-2} + m_{n-1-k,k-1} + m_{n-1-k,k})$$
$$= \left(s_{n-3} - M_{(n-3)/2}\right) + s_{n-2} + s_{n-1}.$$

Case n is even. Clearly, $[n/2] = n/2$. Now we see

$$s_n = \sum_{k=0}^{n/2} m_{n-k,k} = \sum_{k=0}^{n/2} (m_{n-1-k,k-2} + m_{n-1-k,k-1} + m_{n-1-k,k})$$
$$= s_{n-3} + s_{n-2} + s_{n-1}.$$

Figure 2 explains the phenomenon in the odd case with more details.

In the next step, we eliminate the term $M_{(n-3)/2}$ of the Motzkin sequence if n is odd. According to (1.1) we need the coefficients

$$c_0 = \frac{n-3}{2} + 2 = \frac{n+1}{2}, \; c_1 = 2 \cdot \frac{n-3}{2} + 1 = n-2, \; c_2 = 3 \cdot \frac{n-3}{2} - 3 = \frac{3n-15}{2}.$$

Assume that n is odd, and put $\tau_n = M_{(n-3)/2} - s_n + s_{n-1} + s_{n-2} + s_{n-3}$. A straightforward calculation of $c_0\tau_n + c_1\tau_{n-1} + c_2\tau_{n-2}$ provides the statement of the theorem immediately when n is odd.

Suppose now that n is even. Let $d_0 = n+1$, $d_1 = -2n+4$, and $d_2 = -3n+15$. Let $\chi_n = M_{(n-3)/2} - \tau_n$. Similarly to the odd case, the linear combination $d_0\chi_n + d_1\chi_{n-2} + d_2\chi_{n-4}$ leads the recurrence relation of Theorem 2.6 right away. Thus, the proof is complete.

□

Proof of Theorem 2.7. It follows from the definition of the Motzkin triangle and the Motzkin sequence that independently from the parity of n we find $r_n = r_{n-1} + r_{n-2} + (r_n - M_n)$, see Figure 3. Thus we have

$$M_n = r_{n-1} + r_{n-2}.$$

Taking the analogous identity for M_{n-1} and M_{n-2}, and multiplying them by $n+2$, $-(2n+1)$, and $-(3n-3)$, respectively, if we consider finally the sum of them, then it leads immediately to the result

$$(n+3)r_n = nr_{n-1} + (5n+3)r_{n-2} + 3nr_{n-3}.$$

□

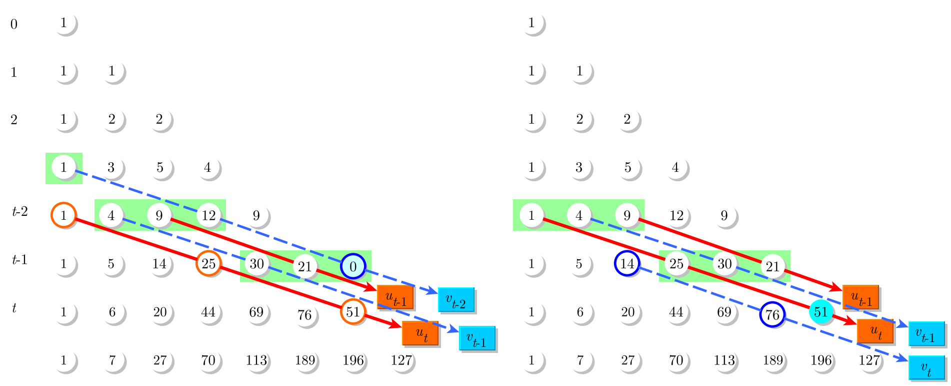

Fig. 5 Padovan descending direction in Motzkin triangle

Proof of Theorem 2.8. It is clear from Figure 5, that the equalities

$$\begin{aligned} u_t &= u_{t-1} + v_{t-1} + v_{t-2}, \\ v_t &= u_t + u_{t-1} + v_{t-1} - M_t \end{aligned} \tag{3.1}$$

hold. The equivalent version

$$\begin{aligned} v_{t-2} &= u_t - u_{t-1} - v_{t-1} \\ v_t + M_t &= u_t + u_{t-1} + v_{t-1} \end{aligned}$$

provides

$$M_t + v_t = u_t + u_{t-1} + u_{t+1} - u_t - v_t,$$

and then

$$M_t + 2v_t = u_{t-1} + u_{t+1} \tag{3.2}$$

follows. According to (1.1), the Motzkin terms disappear from the linear combination

$$-(t+2)(M_t + 2v_t) + (2t+1)(M_{t-1} + 2v_{t-1}) + (3t-3)(M_{t-2} + 2v_{t-2}),$$

and using (3.2) we find the equation

$$\begin{aligned}&-(2t+4)v_t + (4t+2)v_{t-1} + (6t-6)v_{t-2} = \\ &-(t+2)u_{t+1} + (2t+1)u_t + (2t-5)u_{t-1} + (2t+1)u_{t-2} + (3t-3)u_{t-3}\end{aligned}$$

true. Following the straightforward calculations

$$\begin{aligned}&-(2t+4)v_t + (4t+2)v_{t-1} + (6t-6)(u_t - u_{t-1} - v_{t-1}) = \\ &-(t+2)u_{t+1} + (2t+1)u_t + (2t-5)u_{t-1} + (2t+1)u_{t-2} + (3t-3)u_{t-3},\end{aligned}$$

$$\begin{aligned}&-(2t+4)v_t + (-2t+8)v_{t-1} = \\ &-(t+2)u_{t+1} + (-4t+7)u_t + (8t-11)u_{t-1} + (2t+1)u_{t-2} + (3t-3)u_{t-3}\end{aligned}$$

and

$$\begin{aligned}&-(2t+4)v_t + (-2t+8)(u_{t+1} - u_t - v_t) = \\ &-(t+2)u_{t+1} + (-4t+7)u_t + (8t-11)u_{t-1} + (2t+1)u_{t-2} + (3t-3)u_{t-3}\end{aligned}$$

we obtain

$$\begin{aligned}-12v_t = (t-10)u_{t+1} + (-6t+15)u_t + (8t-11)u_{t-1}& \\ +(2t+1)u_{t-2} + (3t-3)u_{t-3}&.\end{aligned} \tag{3.3}$$

Only the left-hand side contains a term from the sequence (v), the right hand side depends on (u). From the first equation of (3.1) we see that $-12v_t - 12v_{t-1} = -12u_{t+1} + 12u_t$. Applying this, together with (3.3), after simplification and shifting the subscripts, finally we have

$$\begin{aligned}(t+1)u_t = (5t+3)u_{t-1} - (2t+8)u_{t-2} - (10t-28)u_{t-3} - (5t-9)u_{t-4}& \\ - (3t-9)u_{t-5}&.\end{aligned}$$

Thus the proof of the first statement is complete. We can obtain the justification of the second statement by using the same method, so we omit the details here, only a short hint. Subtracting the rows of (3.1) from each other we have $2u_t - M_t = v_t + v_{t-2}$. Following the previous calculation (3.3) appears as

$$\begin{aligned}(8t-8)u_t = -(t+2)v_t + (12t-3)v_{t-1} + (18t-15)v_{t-2} + (8t-5)v_{t-3}& \\ + (3t-3)v_{t-4}&,\end{aligned}$$

which implies the result.

□

As an illustration, we notify the first few terms of the sequences (u_n) and (v_n), respectively:

$$(u_t)_{t=0}^{\infty}=\{1,1,2,5,12,30,77,201,532,1424,3847,10474,28707,79133,\ldots\},$$
$$(v_t)_{t=0}^{\infty}=\{0,1,2,5,13,34,90,241,651,1772,4855,13378,37048,103053,\ldots\}.$$

Remark 3.1 The sums given by the ascending Padovan direction can be described by the system

$$\begin{aligned}
p_{3t} &= p_{3t-1} + p_{3t-2} + q_{3t-2} \\
p_{3t-1} &= p_{3t-2} + p_{3t-3} + q_{3t-3} - M_{2t-2} \\
p_{3t-2} &= p_{3t-3} + p_{3t-4} + q_{3t-4} \\
q_{3t} &= p_{3t-1} + q_{3t-1} + q_{3t-2} - M_{2t-1} \\
q_{3t-1} &= p_{3t-2} + q_{3t-2} + q_{3t-3} \\
q_{3t-2} &= p_{3t-3} + q_{3t-3} + q_{3t-4},
\end{aligned}$$

but it is more complicated than the descending case. We may gain a similar result to Theorem 2.8, but the formulation is expected to be rather inconvenient.

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