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How can the product of two binary recurrences be constant?

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abstract: Let ω denote an integer. This paper studies the equation $G_nH_n = \omega$ in the integer binary recurrences $\{G\}$ and $\{H\}$ satisfy the same recurrence relation. The origin of the question gives back to the more general problem $G_nH_n+c = x_{kn+l}$ where c and $k \geq 0$, $l \geq 0$ are fixed integers, and the sequence $\{x\}$ is like $\{G\}$ and $\{H\}$. The case of $k = 2$ has already been solved ([1]) and now we concentrate on the specific case $k = 0$.

Key Words: Binary recurrences.

Contents

1. Introduction

Assume that $A > 0$ and $B \neq 0$ are integers with non-vanishing $D = A^2 + 4B$, further let $\mathcal{B}$ denote the set of all integer binary recurrences $\{X\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$X_{n+2} = AX_{n+1} + BX_n, \qquad n \in \mathbb{N}. \tag{1.1}$$

The equation

$$G_nH_n + c = x_{kn+l}, \qquad n \in \mathbb{N} \tag{1.2}$$

was studied in the case of $k = 2$ in [1], where $\{G\}_{n=0}^{\infty}$, $\{H\}_{n=0}^{\infty}$ and $\{x\}_{n=0}^{\infty}$ belong to the class of binary recurrences given by (1.1), further c is a fixed integer. The reason why $k = 2$ had the main interest is that one expects $G_nH_n \approx x_{2n+l}$ if D is positive. The Fibonacci sequence $\{F\}$ and its companion sequence $\{L\}$ provide a classical example by the identity $F_nL_n = F_{2n}$.

The assumption $k = 0$ is also interesting since c and x_l can be joined in (1.2). In this paper we consider the specific case $k = 0$. There is no restriction in assuming that $l = 0$, since l causes only a translation in the subscript.

2. The equation $G_nH_n = \omega$

For any complex numbers α, β, ... and for any sequence $\{X\}_{n=0}^{\infty} \in \mathcal{B}$ put $\alpha_X = X_1 - \alpha X_0$, $\beta_X = X_1 - \beta X_0$, etc. Recall, that $A > 0$ and $B \neq 0$ are integers and $D = A^2 + 4B \neq 0$.

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Lemma 2.1. *Assume that $\{G\} \in \mathcal{B}$. Then the zeros*

$$\alpha = \frac{A+\sqrt{D}}{2} \quad , \quad \beta = \frac{A-\sqrt{D}}{2} \tag{2.1}$$

of the companion polynomial $p(x) = x^2 - Ax - B$ of $\{G\}$ are distinct. Further,

- $0 \neq \alpha\beta = -B, \quad 0 \neq \alpha+\beta = A, \quad 0 \neq \alpha - \beta = \sqrt{D},$
- $(\alpha^2-1)(\beta^2-1) = (B-1)^2 - A^2,$
- $\alpha \in \mathbb{R}$ *implies* $\beta < \alpha$ *and* $1 < \alpha$. *Especially,* $\alpha^2 \neq 1$.

Moreover,

$$G_n = \frac{\beta_G\alpha^n - \alpha_G\beta^n}{\sqrt{D}}. \tag{2.2}$$

Proof: All formulae and statements of Lemma 2.1 are known. Nevertheless, the first three conditions are immediate from (2.1), while (2.2) can be derived from the basic theorem of the linear recurrences (see, for instance, page 33 in [2]). □

In the sequel, we assume that $c \in \mathbb{Z}$ is given, $l = 0$, further let $\{G\} \in \mathcal{B}$, $\{H\} \in \mathcal{B}$ and $\{x\} \in \mathcal{B}$ satisfy (1.2). Note that α and β are conjugate zeros of $p(x)$ if they are not integers.

In the virtue of (2.2),

$$G_nH_n + c = x_{kn} \tag{2.3}$$

is equivalent to

$$\beta_G\beta_H\alpha^{2n} + \alpha_G\alpha_H\beta^{2n} - (\beta_G\alpha_H + \alpha_G\beta_H)(\alpha\beta)^n + Dc = \sqrt{D}(\beta_x\alpha^{kn} - \alpha_x\beta^{kn}), \tag{2.4}$$

and by $k = 0$ the right hand side of (2.4) comes up with $\sqrt{D}(\beta_x - \alpha_x) = Dx_0$. Put $\delta = D(x_0 - c)$. Then we obtain

$$\beta_G\beta_H\alpha^{2n} + \alpha_G\alpha_H\beta^{2n} - (\beta_G\alpha_H + \alpha_G\beta_H)(\alpha\beta)^n - \delta = 0. \tag{2.5}$$

Obviously, (2.5) is true for all n, therefore it is true for $n = 0,\ 1,\ 2,\ 3$. Consequently, (2.5) at $n = 0,\ 1,\ 2,\ 3$ is a homogeneous linear system of four equations in the unknowns $\beta_G\beta_H$, $\alpha_G\alpha_H$, $(\beta_G\alpha_H + \alpha_G\beta_H)$ and δ. The determinant

$$\mathcal{D} = \begin{vmatrix} 1 & 1 & -1 & -1 \\ \alpha^2 & \beta^2 & -\alpha\beta & -1 \\ \alpha^4 & \beta^4 & -(\alpha\beta)^2 & -1 \\ \alpha^6 & \beta^6 & -(\alpha\beta)^3 & -1 \end{vmatrix}$$

of the coefficient matrix is the Vandermonde of α^2, β^2, $\alpha\beta$ and 1. Hence

$$\mathcal{D} = V(\alpha^2, \beta^2, \alpha\beta, 1) = -\alpha\beta(\alpha-\beta)^3(\alpha+\beta)(\alpha^2-1)(\beta^2-1)(\alpha\beta-1). \tag{2.6}$$

CASE I. Suppose first, that $\mathcal{D} \neq 0$, i.e. the homogeneous system has only the trivial solution

$$\begin{aligned} 0 &= \beta_G\beta_H = (G_1-\beta G_0)(H_1-\beta H_0)\\ 0 &= \alpha_G\alpha_H = (G_1-\alpha G_0)(H_1-\alpha H_0)\\ 0 &= \beta_G\alpha_H+\alpha_G\beta_H = (G_1-\beta G_0)(H_1-\alpha H_0)+(G_1-\alpha G_0)(H_1-\beta H_0)\\ 0 &= \delta = D(x_0-c) \end{aligned}$$

The last equality shows that $x_0 = c$, consequently (2.3) implies $G_nH_n = 0$ for all non-negative integers n. If $\{G\}$ or $\{H\}$ is the zero sequence then G_nH_n vanishes. Therefore we suppose that neither $\{G\}$ nor $\{H\}$ is the zero sequence.

First assume $G_1-\beta G_0 = 0$, such that $G_1 \neq 0$ and $G_0 \neq 0$ (otherwise $\{G\}$ would be the zero sequence). Thus $\alpha \neq \beta$ implies $G_1-\alpha G_0 \neq 0$, hence we have $H_1-\alpha H_0 = 0$ and $H_1-\beta H_0 = 0$. Consequently, $H_1 = H_0 = 0$ and we arrived at a contradiction.

If one begins with $H_1-\beta H_0 = 0$, it similarly leads to a contradiction. Thus we get, that in Case I at least one of the sequences $\{G\}$ and $\{H\}$ must be the constant zero sequence.

CASE II. If the system of four equation has no unique solutions (i.e. $\mathcal{D} = 0$), then all the infinitely many solutions can be given by

$$\beta_G\beta_H = \frac{(\beta^2-1)(\alpha\beta-1)}{\alpha(\alpha-\beta)(\alpha^2-\beta^2)}\,\delta,\quad \alpha_G\alpha_H = \frac{(\alpha^2-1)(\alpha\beta-1)}{\beta(\alpha-\beta)(\alpha^2-\beta^2)}\,\delta,$$
$$\beta_G\alpha_H+\alpha_G\beta_H = \frac{(\alpha^2-1)(\beta^2-1)}{\alpha\beta(\alpha-\beta)^2}\,\delta, \tag{2.7}$$

where $\delta = D(x_0-c)$ is a free parameter.

Recalling (2.6) and the conditions $\alpha\beta \neq 0$, $\alpha \neq \beta$, $\alpha+\beta \neq 0$, $\alpha^2 \neq 1$ (see Lemma 2.1), we may distinguish three branches.

1. $\beta-1 = 0$**.** Now $\beta = 1$ implies $\alpha = A-1$, $B = 1-A$ and $D = (A-2)^2$. Thus we have $X_n = AX_{n-1}+(1-A)X_{n-2}$ as the recurrence rule of $\mathcal{B}$. The solutions in (2.7) simplify to

$$\begin{aligned} 0 &= \beta_G\beta_H = (G_1-G_0)(H_1-H_0),\\ \delta = D(x_0-c) &= \alpha_G\alpha_H = (G_1-(A-1)G_0)\,(H_1-(A-1)H_0)\,, \qquad (2.8)\\ 0 &= \beta_G\alpha_H+\alpha_G\beta_H = (G_1-G_0)\,(H_1-(A-1)H_0)\\ &\qquad +(G_1-(A-1)G_0)\,(H_1-H_0). \end{aligned}$$

If $G_1 = G_0$ then $\{G\}$ is a constant sequence. It is either the constant zero sequence or $H_1 = H_0$ holds and we conclude that $\{H\}$ is also constant. Subsequently, (2.8) becomes $D(x_0-c) = (2-A)^2G_0H_0$, and then $G_0H_0+c = x_0$ follows.

Note, that starting with the symmetric assumption $H_1 = H_0$, it also leads to the same conclusion.

2. $\beta+1=0$**.** Here we follow the treatment of the case of $\beta=1$. Now $\beta=-1$ provides $\alpha=B=A+1$, $D=(A+2)^2$ and $X_n=AX_{n-1}+(A+1)X_{n-2}$. Hence (2.7) reduces to

$$\begin{aligned} 0 &= \beta_G\beta_H=(G_1+G_0)(H_1+H_0), \\ \delta=D(x_0-c) &= \alpha_G\alpha_H=(G_1-(A+1)G_0)\left(H_1-(A+1)H_0\right), \\ 0 &= \beta_G\alpha_H+\alpha_G\beta_H=(G_1+G_0)\left(H_1-(A+1)H_0\right) \\ &\qquad +\left(G_1-(A+1)G_0\right)(H_1+H_0). \end{aligned}$$

Supposing $G_1=-G_0$ we obtain that either $\{G\}$ is the constant zero sequence, or it is an alternate sequence of G_0 and $-G_0$. In the latter case $\{H\}$ is also alternate of H_0 and $-H_0$.

3. $\alpha\beta-1=0$**.** It is easy to see that neither α nor β is rational. Obviously, $B=-1$ and $X_n=AX_{n-1}-X_{n-2}$. Moreover, $D=A^2-4\neq 0$ yields $A\neq 2$. Now the formulae in (2.7) become

$$\begin{aligned} 0 &= \beta_G\beta_H=(G_1-\beta G_0)(H_1-\beta H_0), \\ 0 &= \alpha_G\alpha_H=(G_1-\alpha G_0)\left(H_1-\alpha H_0\right), \\ -\delta=-D(x_0-c) &= \beta_G\alpha_H+\alpha_G\beta_H=(G_1-\beta G_0)\left(H_1-\alpha H_0\right) \\ &\qquad +(G_1-\alpha G_0)\left(H_1-\beta H_0\right). \end{aligned}$$

The first equation provides, for example $G_1=G_0=0$, since $\beta\notin\mathbb{Q}$. Hence $\delta=0$ and $x_0=c$.

The observations above prove the following theorem.

Theorem 2.2. *Suppose that for all natural number n the terms G_n and H_n of sequences $\{G\}$ and $\{H\}$, respectively, satisfy the equality*

$$G_nH_n+c=x_0.$$

Then one of the following three cases holds.

- *Either $\{G\}$ or $\{H\}$ is the constant zero sequence, and $x_0=c$.*
- *Both $\{G\}$ and $\{H\}$ are constant sequences, and $x_0=G_0H_0+c$.*
- *Both $\{G\}$ and $\{H\}$ are alternate sequences given by $G_n=(-1)^nG_0$ and $H_n=(-1)^nH_0$, and $x_0=G_0H_0+c$.*

The first option is possible for arbitrary recurrence class $\mathcal{B}$. The second case requires $\beta=1$, while the third $\beta=-1$.

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References

1. Khadir, O. – Liptai, K. – Szalay, L., On the shifted product of binary recurrences, Journal of Integer Sequences, Vol. 13 (2010), Article 10.6.1.

2. T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge University Press 1986.

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